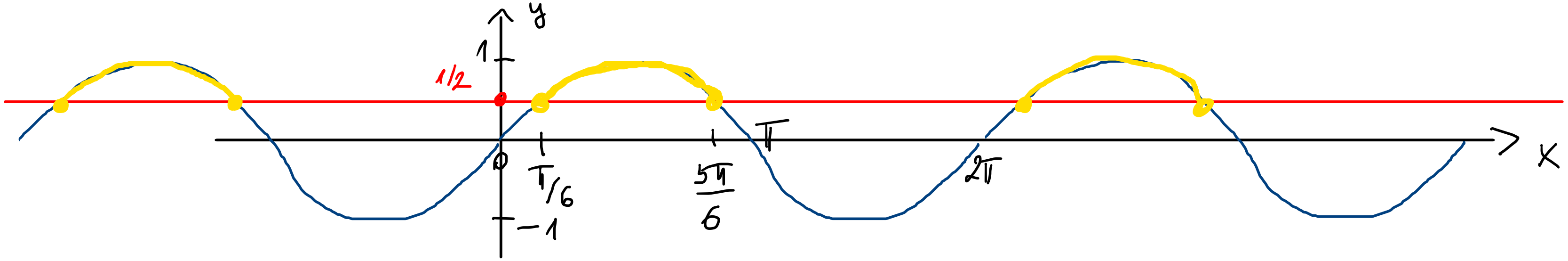
Mintavizsga 2. - Mat. A1a, GTK, 2024.

① $\sin(2x-1) \geqq \frac{1}{2}$ $\quad x \in \mathbb{R}$ $\quad$ 1p



$$\sin(2x-1) \geqq \frac{1}{2}$$

$$\Updownarrow$$

$$\frac{\pi}{6} + k \cdot 2\pi \leqq 2x - 1 \leqq \frac{5\pi}{6} + k 2\pi \quad \Big/ +1 \text{ majd } : 2 \qquad 2p$$

$$k \in \mathbb{Z} \qquad 1p$$

$$\frac{1 + \frac{\pi}{6} + k \cdot 2\pi}{2} \leqq x \leqq \frac{1 + \frac{5\pi}{6} + k \cdot 2\pi}{2} \qquad 2p$$

(2) $$\underbrace{\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}}_{B.O.} = \frac{1}{2}\cdot\underbrace{\frac{\left(\frac{1}{2}\right)^n-1}{\frac{1}{2}-1}}$$

$$= -\left(\left(\frac{1}{2}\right)^n-1\right) = 1-\left(\frac{1}{2}\right)^n \quad J.O.$$

1] $n=1$: $\underset{B.O.}{\frac{1}{2}} = \underset{J.O.}{1-\frac{1}{2}}$ ✓ $\qquad n=2$: $\underset{B.O.}{\frac{1}{2}+\frac{1}{4}=\frac{3}{4}} = \underset{J.O.}{1-\frac{1}{2^2}} = \frac{3}{4}$ ✓ (2p)

2] Tfh. $\forall$ term. számra igaz $n=m$-ig az állítás:

$$\sum_{k=1}^{m}\frac{1}{2^k} = 1-\left(\frac{1}{2}\right)^m$$ indukciós feltevés (2p)

3] Ellenőrzés $n=m+1$-re:

$$B.O. = \sum_{k=1}^{m+1}\frac{1}{2^k} = \left(\sum_{k=1}^{m}\frac{1}{2^k}\right)+\frac{1}{2^{m+1}} \overset{\text{ind. felt.}}{=} 1-\left(\frac{1}{2}\right)^m+\frac{1}{2^{m+1}} = 1-\left(\frac{1}{2}\right)^m+\left(\frac{1}{2}\right)^{m+1} =$$

$$= 1-1\cdot\left(\frac{1}{2}\right)^m+\frac{1}{2}\cdot\left(\frac{1}{2}\right)^m = 1-\left(\underbrace{1-\frac{1}{2}}_{1/2}\right)\cdot\left(\frac{1}{2}\right)^m = 1-\left(\frac{1}{2}\right)^{m+1} \equiv J.O.$$ ✓ (3p)

Azonosságot kaptunk, tehát az eredeti állítás minden pozitív természetes számra igaz. (1p)

3

$20;\ -10;\ 5;\ -2{,}5;\ \dots$

$a_1 \quad a_2 \quad a_3 \quad a_4$

$$\frac{a_2}{a_1} = \frac{-1}{2} = \frac{a_3}{a_2} = \frac{a_4}{a_3} \Rightarrow q = \frac{-1}{2}$$ 3p

$$a_5 = a_4 \cdot q = -2{,}5 \cdot \frac{-1}{2} = +\frac{5}{2} \cdot \frac{1}{2} = \frac{5}{4}$$ 2p

$$a_n = a_1 \cdot q^{n-1} = 20 \cdot \left(\frac{-1}{2}\right)^{n-1}$$ 3p

(4) $f(x)=\dfrac{x-1}{e^x}$

1) $e^x>0 \quad \forall x\in\mathbb{R} \Rightarrow D_f=\mathbb{R}$

2) zh: $x-1=0 \iff x=1$ (1p)

3) nem periodikus, és mivel a zh-ek elhelyezkedése nem origóra szimm., így nem páros és nem ptlan a fgv.

4) $\lim\limits_{x\to-\infty}\dfrac{x-1}{e^x} = -\infty$ (1p), " $\dfrac{-\infty}{0^{+}}$ " h'pus

ill $\lim\limits_{x\to\infty}\dfrac{x-1}{e^x}$ " $\dfrac{\infty}{\infty}$ " $\underset{\text{L'HOSPITAL}}{=} \lim\limits_{x\to\infty}\dfrac{(x-1)'}{(e^x)'} = \lim\limits_{x\to\infty}\dfrac{1}{e^x}=0$ " $\dfrac{1}{\infty}$ " (1p)

($+\infty$-ben vízszintes aszimptota: x-teng.)

Ferde aszimptota $-\infty$-ben?

$m=\lim\limits_{x\to-\infty}\dfrac{f(x)}{x} = \lim\limits_{x\to-\infty}\underbrace{\dfrac{x-1}{x}}_{\to 1}\cdot\underbrace{\dfrac{1}{e^x}}_{\to+\infty} = 1\cdot\infty=\infty \Rightarrow$ nincs ferde aszimptota (1p)

5) $f'(x)=\dfrac{1\cdot e^x-(x-1)e^x}{(e^x)^2}=\dfrac{(-x+2)\cancel{e^x}}{(e^x)^{\cancel{2}}}$

	$x<2$	$x=2$	$x>2$
f'	+	0	—
f	szig. mon. nő ↗	Lok. MAX	szig. mon. csökk. ↘

(2p)

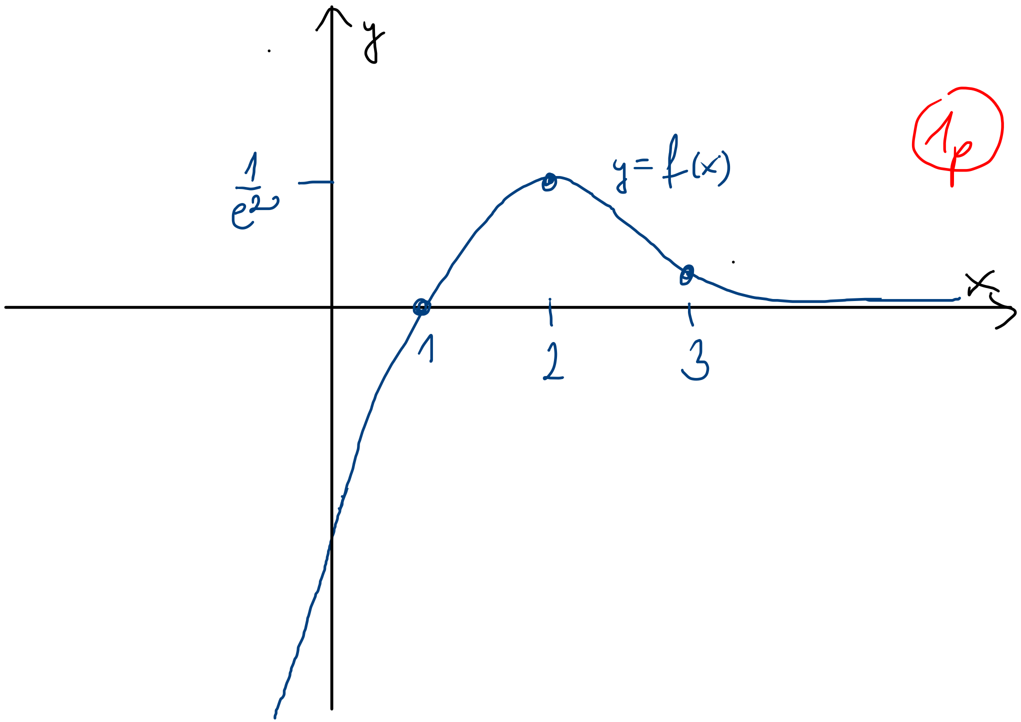
$f(2)=\dfrac{1}{e^2}$

6) $f''(x) = \left(\dfrac{-x+2}{e^x}\right)' = \dfrac{-\cancel{e^x} - (-x+2)\cancel{e^x}}{(e^x)^{\cancel{2}}} = \dfrac{x-3}{e^x}$

	$x<3$	$x=3$	$x>3$
f''	$-$	0	$+$
f	konkáv ⌒	IP	konvex ⌣

2p

7)



1p

8) $\mathcal{D}_f = \left]-\infty\,;\,\frac{1}{e^2}\right]$

1p

5

a) $\int \underset{g'}{1} \cdot \underset{f}{\ln(x^2)}\,dx = x \cdot \ln(x^2) - \int \underbrace{x \cdot \frac{2}{x}}_{=2}\,dx = x \cdot \ln(x^2) - 2x + C$ 1p

1p

$g = x \qquad f' = \frac{2x}{x^2} = \frac{2}{x}$ 1p

$C \in \,]-\infty; \infty[$ 1p

b) $\int (5x^3+30x)^4 \cdot (x^2+2)\,dx = \frac{1}{15} \cdot \int \underbrace{(5x^3+30x)^4}_{f(g(x))} \cdot \underbrace{15 \cdot (x^2+2)}_{g'(x)}\,dx =$ 2p

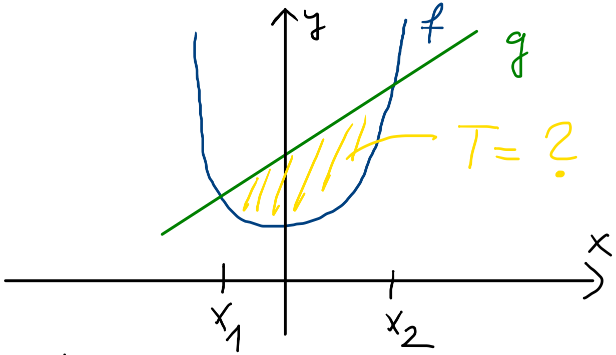
$(5x^3+30x)' = 15x^2+30 = 15 \cdot (x^2+2)$

$= \frac{1}{15} \cdot \frac{(5x^3+30x)^5}{5} + C$ 1p

$C \in \mathbb{R}$ 1p

⑥ $f(x) = 2x^2 + 1$

$g(x) = 7x + 5$



$2x^2 + 1 = 7x + 5$ (1p)

$2x^2 - 7x - 4 = 0$

$$x_{1,2} = \frac{7 \pm \sqrt{\overset{9}{49 + 32}}}{4} = \begin{cases} x_1 = -\frac{1}{2} \\ x_2 = 4 \end{cases}$$ (2p)

$$T = \int_{x=x_1}^{x_2} g(x) - f(x)\,dx = \int_{x=-\frac{1}{2}}^{4} 7x + \underbrace{5 - 2x^2 - 1}_{-2x^2 + 7x + 4}\,dx = \left[\frac{-2x^3}{3} + \frac{7x^2}{2} + 4x\right]_{-\frac{1}{2}}^{4} =$$

(1p) (1p) (2p)

$$= \frac{-128}{3} + 56 + 16 - \left(\frac{+2}{24} + \frac{7}{8} - 2\right)$$ (1p)