EXPONENCIÁLIS MÁTRIX - FÜGGVÉNY

- Definíció
- Kiszámítás(ok)
- Alkalmazás(ok)

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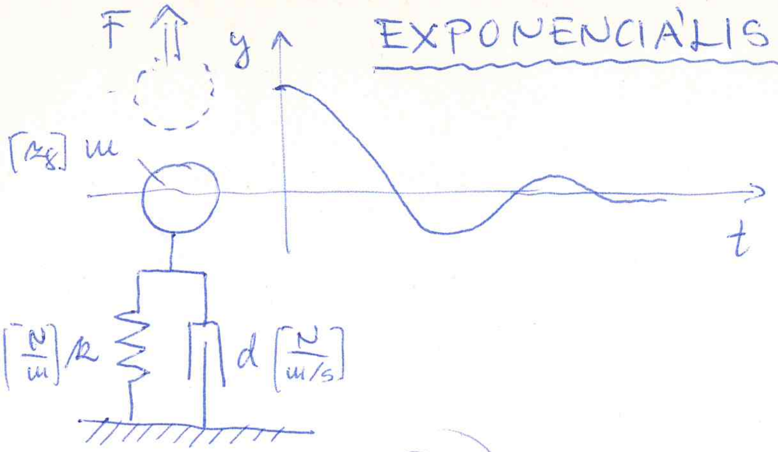
KNOPP FERENC (BME-MOGI)

knopp@mogi.bme.hu

06/70/580-7324

EXPONENCIÁLIS MÁTRIXFÜGGVÉNY



NEWTON II.

$$m\cdot\ddot{y}(t) = -k\cdot y(t) - d\cdot\dot{y}(t) + F(t)$$

$$\boxed{m\cdot\ddot{y}(t) + d\cdot\underbrace{\dot{y}(t)}_{x_2} + k\cdot\underbrace{y(t)}_{x_1} = 0} \quad = F(t)$$

$F(t) = 0$

$x_1 = y$

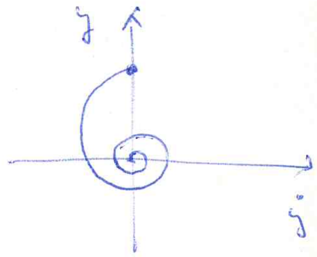
$x_2 = \dot{y} = \dot{x}_1$

$\dot{x}_2 = \ddot{y} = -\frac{k}{m}\cdot y - \frac{d}{m}\cdot\dot{y} = -\frac{k}{m}\cdot x_1 - \frac{d}{m}\cdot x_2 + \frac{1}{m}\cdot F$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{d}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}\cdot u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\end{bmatrix}\cdot u$$

$(u = F)$



$m = 1$ [kg]

$k = 2$ [N/m] (Lágy!)

$d = 3$ [N/(m/s)]

$F = u = 0$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

HOMOGÉN eset

$$\boxed{\begin{aligned}\dot{\underline{x}}(t) &= \underline{\underline{A}}\cdot\underline{x}(t)\\ \underline{x}(0) &= \underline{x}_0\end{aligned}}$$

TAYLOR sor $t=0$ körül (Feltevés!)

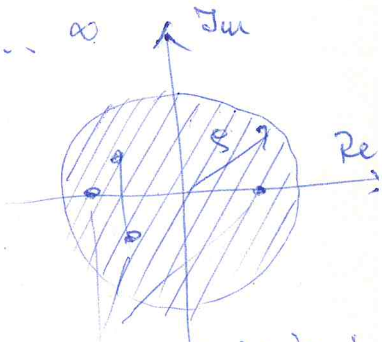
$$\underline{x}(t) = \underline{x}(0) + \dot{\underline{x}}(0)\cdot t + \ddot{\underline{x}}(0)\cdot\frac{t^2}{2!} + \dddot{\underline{x}}(t)\cdot\frac{t^3}{3!} + \dots \infty$$

$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} \longrightarrow \dot{\underline{x}}(0) = \underline{\underline{A}}\cdot\underline{x}(0)$

$\ddot{\underline{x}} = \underline{\underline{A}}\cdot\dot{\underline{x}} = \underline{\underline{A}}(\underline{\underline{A}}\,\underline{x}) = \underline{\underline{A}}^2\cdot\underline{x} \longrightarrow \ddot{\underline{x}}(0) = \underline{\underline{A}}^2\cdot\underline{x}(0)$

$\dddot{\underline{x}} = \underline{\underline{A}}^2\cdot\dot{\underline{x}} = \underline{\underline{A}}^2(\underline{\underline{A}}\cdot\underline{x}) = \underline{\underline{A}}^3\cdot\underline{x} \longrightarrow \dddot{\underline{x}}(0) = \underline{\underline{A}}^3\cdot\underline{x}(0)$

$\vdots$



$\underline{\underline{A}}$ sajátértékei

$$\underline{x}(t) = \Big[\underbrace{\underline{\underline{I}} + \underline{\underline{A}}\cdot t + \underline{\underline{A}}^2\cdot\frac{t^2}{2!} + \underline{\underline{A}}^3\cdot\frac{t^3}{3!} + \dots \infty}\Big]\cdot\underline{x}(0)$$

DEF. $e^{\underline{\underline{A}}t} = \exp(\underline{\underline{A}}t)$

Mindig KONVERGENS!!!

$$\boxed{\underline{x}(t) = e^{\underline{\underline{A}}t}\cdot\underline{x}(0)}$$

$$\rho = \frac{1}{\limsup\limits_{n\to\infty}\sqrt[n]{|c_n|}} = \infty$$

az e^z

1/a

$$\dot{x}(t) = a \cdot x(t) \longrightarrow \frac{dx}{x} = a \cdot dt \quad \ldots \quad \ln|x| = a \cdot t + \ln C$$

$$\ln \frac{x}{C} = at$$

$$\boxed{x(t) = C \cdot e^{at}}$$

$$x(0) = C$$

TAYLOR sor $(t_0 = 0)$

$$x(t) = x(0) + \dot{x}(0) \cdot t + \frac{t^2}{2!} \cdot \ddot{x}(0) \cdot + \ldots$$

$$\dot{x} = ax$$

$$\ddot{x} = a\dot{x} = a \cdot (ax) = a^2 \cdot x$$

$$\vdots$$

$$x(t) = x(0) + a \cdot x(0) \cdot t + \frac{t^2}{2!} \cdot a^2 \cdot x(0) + \ldots =$$

$$= x(0) \Big[\underbrace{1 + (at) + \frac{(at)^2}{2!} + \ldots \infty}_{e^{at}} \Big] = e^{at} \cdot x(0)$$

MEGJEGYZÉS: Szukcesszív approximációval

$$\underline{x}_{k+1}(t) = \underbrace{\underline{x}_0(t)}_{\underline{x}(0)} + \int_0^t \underline{f}(\underline{x}_k(\tau))\,d\tau \qquad \dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} = \underline{f}(\underline{x})$$

$$\underline{x}_0(t) = \underline{x}(0)$$

$$\underline{x}_1(t) = \underline{x}(0) + \int_0^t \underline{\underline{A}}\cdot\underline{x}(0)\cdot d\tau = \underline{x}(0) + \underline{\underline{A}}\cdot t\cdot\underline{x}(0)$$

$$\underline{x}_2(t) = \underline{x}(0) + \int_0^t \left[\underline{\underline{A}}\left(\underline{x}(0) + \underline{\underline{A}}\cdot\underline{x}(0)\cdot\tau\right)\right]d\tau = \underline{x}(0) + \underline{\underline{A}}t\underline{x}(0) + \underline{\underline{A}}^2\cdot\frac{t^2}{2}\cdot\underline{x}(0)$$

$$\underline{x}_3(t) = \Big[\underbrace{\underline{\underline{I}} + (\underline{\underline{A}}t) + \frac{1}{2!}(\underline{\underline{A}}t)^2 + \frac{1}{3!}(\underline{\underline{A}}t)^3}_{\text{(szelete)}\ \to\ e^{\underline{\underline{A}}t}}\Big]\cdot\underline{x}(0)$$

$$\vdots$$

VÉGES ELŐÁLLÍTÁS $\quad \underline{\underline{A}} \in \mathbb{R}^{n\times n}$ (CAYLEY-HAMILTON tétel alapján)

$$(1)\quad e^{\underline{\underline{A}}t} = \underline{\underline{I}} + (\underline{\underline{A}}t) + \frac{1}{2!}(\underline{\underline{A}}t)^2 + \frac{1}{3!}(\underline{\underline{A}}t)^3 + \dots \infty \qquad /\cdot\underline{s}$$

$$(2)\quad e^{\underline{\underline{A}}t} = \gamma_0(t)\cdot\underline{\underline{I}} + \gamma_1(t)\cdot\underline{\underline{A}} + \gamma_2(t)\cdot\underline{\underline{A}}^2 + \dots + \gamma_{n-1}(t)\cdot\underline{\underline{A}}^{n-1} \qquad /\cdot\underline{s}$$

$\underline{\underline{A}}\cdot\underline{s} = \lambda\cdot\underline{s}$ sajátvektor

$$\underline{\underline{A}}^2\cdot\underline{s} = \underline{\underline{A}}(\underline{\underline{A}}\cdot\underline{s}) = \underline{\underline{A}}\cdot(\lambda\cdot\underline{s}) = \lambda(\underline{\underline{A}}\cdot\underline{s}) = \lambda^2\cdot\underline{s}$$

$$\vdots$$

$$\underline{\underline{A}}^k\cdot\underline{s} = \lambda^k\cdot\underline{s}$$

$$\vdots$$

$$(1)\quad e^{\underline{\underline{A}}t}\cdot\underline{s} = \Big[\overbrace{1 + (\lambda t) + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \dots\infty}^{e^{\lambda t}}\Big]\cdot\underline{s}$$

$$(2)\quad e^{\underline{\underline{A}}t}\cdot\underline{s} = \left[\gamma_0(t) + \gamma_1(t)\cdot\lambda + \gamma_2(t)\cdot\lambda^2 + \dots + \gamma_{n-1}(t)\cdot\lambda^{n-1}\right]\cdot\underline{s}$$

Különböző sajátértékek esetén ($\lambda_i \neq \lambda_j$ ha $i \neq j$)

$$\underbrace{\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix}}_{\underline{\underline{V}}} \begin{bmatrix} \gamma_0(t) \\ \gamma_1(t) \\ \vdots \\ \gamma_{n-1}(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

Lineárisan független függvények !!!

$\underline{\underline{V}}$ Vandermonde mátrix $\quad \det\underline{\underline{V}} \neq 0$

$$\underline{\underline{V}}\cdot\underline{\gamma}(t) = \underline{e}(t) \longrightarrow \boxed{\underline{\gamma}(t) = \underline{\underline{V}}^{-1}\cdot\underline{e}(t)}$$

ők is lineárisan függetlenek.

Példa: $\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ (Előző mechanikai lengő-rendszer)

$$\underline{\underline{A}} \cdot \underline{s} = \lambda \underline{s} \Rightarrow \underbrace{(\lambda \underline{\underline{I}} - \underline{\underline{A}})}_{\text{szinguláris legyen!}} \cdot \underline{s} = \underline{0} \Rightarrow \det(\lambda \underline{\underline{I}} - \underline{\underline{A}}) = 0 = D(\lambda)$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix}$$

$$D(\lambda) = \lambda(\lambda+3)+2 = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$$

$$\left.\begin{matrix} \lambda_1 = -1 \\ \lambda_2 = -2 \end{matrix}\right\} \text{ Stabil rendszer!}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \gamma_0(t) \\ \gamma_1(t) \end{bmatrix} = \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix}$$

CRAMER szabállyal:

$$\boxed{\gamma_0(t)} = \frac{\begin{vmatrix} e^{-t} & -1 \\ e^{-2t} & -2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix}} = \frac{-2e^{-t} + e^{-2t}}{-1} = \boxed{2e^{-t} - e^{-2t}}$$

$$\boxed{\gamma_1(t)} = \frac{\begin{vmatrix} 1 & e^{-t} \\ 1 & e^{-2t} \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix}} = \frac{e^{-2t} - e^{-t}}{-1} = \boxed{e^{-t} - e^{-2t}}$$

$$e^{\underline{\underline{A}}t} = (2e^{-t} - e^{-2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t}) \cdot \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$e^{\underline{\underline{A}}t} = \left[\begin{array}{c:c} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ \hdashline 2(e^{-2t} - e^{-t}) & 2e^{-2t} - e^{-t} \end{array}\right] = \underline{\underline{\Phi}}(t) = \left[\begin{array}{c:c} \varphi_{11}(t) & \varphi_{12}(t) \\ \hdashline \varphi_{21}(t) & \varphi_{22}(t) \end{array}\right]$$

NUMERIKUS SZÁMÍTÁSA

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \varphi_{11}(t) & \varphi_{12}(t) & \cdots & \varphi_{1n}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) & \cdots & \varphi_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \varphi_{n1}(t) & \varphi_{n2}(t) & \cdots & \varphi_{nn}(t) \end{bmatrix}}_{e^{\underline{\underline{A}}t}} \begin{bmatrix} \boxed{1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ \boxed{1} \\ \vdots \\ 0 \end{bmatrix}$$

Egységvektorokból indítva a (numerikus) megoldást, megkapjuk $e^{\underline{\underline{A}}t}$ megfelelő oszlopait.

$$c_1 \cdot Mx_1 + \ldots + c_n \cdot Mx_n = 0$$

$$M(c_1 x_1 + \ldots + c_n x_n) = 0$$

ÁLLÍTÁS

Akkor ezek is

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ m_{21} & m_{22} & \ldots & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & \ldots & m_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Ha ezek lineárisan függetlenek

$$\left.\begin{aligned} y_1 &= (m_{11} \cdot x_1 + \ldots + m_{1n} x_n) \\ &\vdots \\ y_n &= (m_{n1} \cdot x_1 + \ldots + m_{nn} \cdot x_n) \end{aligned}\right\}$$

$$\boxed{c_1 y_1 + \ldots + c_n y_n = 0} =$$

$$= c_1 (m_{11} x_1 + \ldots + m_{1n} \cdot x_n) + \ldots + c_n (m_{n1} \cdot x_1 + \ldots + m_{nn} \cdot x_n) =$$

$$= \underbrace{(c_1 m_{11} + \ldots + c_n m_{n1})}_{0} \cdot x_1 + \ldots + \underbrace{(c_1 m_{1n} + \ldots + c_n m_{nn})}_{0} \cdot x_n$$

$$\begin{bmatrix} m_{11} & \ldots & m_{n1} \\ \vdots & & \vdots \\ m_{1n} & \ldots & m_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underline{\underline{M}} \cdot \underline{c} = \underline{0} \longrightarrow \underline{c} = \underline{\underline{M}}^{-1} \cdot \underline{0} = \underline{0}$$

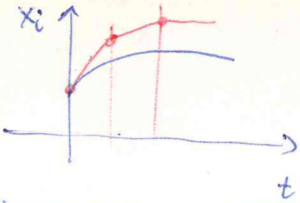
$$\boxed{c_1 = c_2 = \ldots = c_n = 0}$$

$$\| \underline{\underline{A}}\, \underline{x} \| = \sqrt{(a_{11} x_1 + \ldots + a_{1n} x_n)^2 + \ldots + (a_{n1} x_1 + \ldots + a_{nn} x_n)^2} =$$

$$= \sqrt{}$$

Schwartz: $\quad |a_1 b_1 + \ldots + a_n b_n|^2 \leq (|a_1|^2 + \ldots)(|b_1|^2 + \ldots)$

EULER módszer



$$\underline{\dot{x}}(t) = \underline{f}(t, \underline{x}(t), \underline{u}(t))$$

$$\underline{\dot{x}}(t) \simeq \frac{\underline{x}(t+\Delta t) - \underline{x}(t)}{\Delta t} \longrightarrow \boxed{\underline{x}(t+\Delta t) \cong \underline{x}(t) + \Delta t \cdot \underline{\dot{x}}(t)}$$

$\underline{f}(\dots)$

RUNGE-KUTTA
1. Egyszerű (EULER)
2.
3.
4. Elég pontos

INHOMOGÉN eset

$$\boxed{\underline{\dot{x}}(t) = \underline{\underline{A}} \cdot \underline{x}(t) + \underline{\underline{B}} \cdot \underline{u}(t)} \qquad \underline{x}(0) = \underline{x}_0$$

Homogén általános: $\underline{x}_h(t) = e^{\underline{\underline{A}} \cdot t} \cdot \underline{x}_0$

Konstansok

Partikuláris:

$$\underline{x}_p(t) = e^{\underline{\underline{A}} t} \cdot \underline{c}(t)$$

Konstansok variálása

$$\underline{\dot{x}}_p = \cancel{\underline{\underline{A}} e^{\underline{\underline{A}} t} \cdot \underline{c}} + e^{\underline{\underline{A}} t} \cdot \underline{\dot{c}} = \cancel{\underline{\underline{A}} e^{\underline{\underline{A}} t} \cdot \underline{c}} + \underline{\underline{B}} \cdot \underline{u}$$

$$e^{-\underline{\underline{A}} t} / \quad e^{\underline{\underline{A}} t} \cdot \underline{\dot{c}} = \underline{\underline{B}} \cdot \underline{u}$$

$$\underline{\dot{c}}(t) = e^{-\underline{\underline{A}} t} \cdot \underline{\underline{B}} \cdot \underline{u}(t) \longrightarrow \underline{c}(t) = \int_0^t e^{-\underline{\underline{A}} \tau} \cdot \underline{\underline{B}} \cdot \underline{u}(\tau)\, d\tau$$

Teljes megoldás

$$\underline{x}(t) = \underline{x}_h(t) + \underline{x}_p(t) = e^{\underline{\underline{A}} t} \cdot \underline{x}_0 + e^{\underline{\underline{A}} t} \cdot \int_0^t e^{-\underline{\underline{A}} \tau} \cdot \underline{\underline{B}} \cdot \underline{u}(\tau)\, d\tau$$

Bevihető

$$\boxed{\underline{x}(t) = e^{\underline{\underline{A}} t} \cdot \underline{x}_0 + \int_0^t e^{\underline{\underline{A}}(t-\tau)} \cdot \underline{\underline{B}} \cdot \underline{u}(\tau) \cdot d\tau}$$

"Megoldó képlet"

$$\gamma_0(\tau) \cdot \underline{\underline{I}} + \gamma_1(\tau) \cdot \underline{\underline{A}} + \dots + \gamma_{n-1}(\tau) \cdot \underline{\underline{A}}^{n-1}$$

~~3/1~~ 4/a

RUNGE-KUTTA módszerek

R-K 2 $\quad \underline{k}_1 = \underline{f}(t, \underline{x}(t)) \cdot h \qquad h = \Delta t$

$$\underline{k}_2 = \underline{f}(t+h, \underline{x}(t) + \underline{k}_1) \cdot h$$

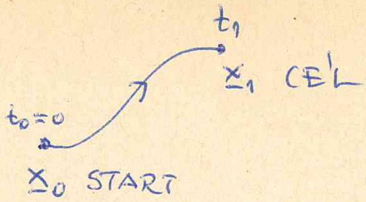
$$\boxed{\underline{x}(t+h) \cong \underline{x}(t) + \frac{1}{2}(\underline{k}_1 + \underline{k}_2)}$$

$\vdots$

$\underline{k}_1 = \qquad \underline{k}_2 = \qquad \underline{k}_3 = \qquad \underline{k}_4 =$

4 $\quad \boxed{\underline{x}(t+h) \cong \underline{x}(t) + \frac{1}{6}(\underline{k}_1 + 2\underline{k}_2 + 2\underline{k}_3 + \underline{k}_4)}$

IRÁNYÍTHATÓSÁG



$$\dot{\underline{x}}(t) = \underline{\underline{A}}\cdot\underline{x}(t) + \underline{\underline{B}}\cdot\underline{u}(t)$$

$$\underline{x}_1 = \underbrace{e^{\underline{\underline{A}}t_1}\cdot\underline{x}_0}_{\text{saját mozgás}} + \int_0^{t_1} e^{\underline{\underline{A}}(t_1-\tau)}\cdot\underline{\underline{B}}\cdot\underline{u}(\tau)\cdot d\tau$$

$$\underbrace{\underline{x}_1 - e^{\underline{\underline{A}}t_1}\cdot\underline{x}_0}_{\underline{z}} = \int_0^{t_1}\left[\gamma_0\cdot\underline{\underline{I}} + \gamma_1(\tau)\cdot\underline{\underline{A}} + \ldots + \gamma_{n-1}(\tau)\cdot\underline{\underline{A}}^{n-1}\right]\cdot\underline{\underline{B}}\cdot\underline{u}(\tau)\,d\tau$$

$$\underline{z} = \underbrace{\left[\underline{\underline{B}} \,\vdots\, \underline{\underline{A}}\,\underline{\underline{B}} \,\vdots\, \ldots\ldots \,\vdots\, \underline{\underline{A}}^{n-1}\underline{\underline{B}}\right]}_{\underline{\underline{Q}}}\begin{bmatrix} \int\gamma_0(\tau)\cdot\underline{u}(\tau) \\ \int\gamma_1(\tau)\cdot\underline{u}(\tau) \\ \vdots \\ \int\gamma_{n-1}(\tau)\cdot\underline{u}(\tau) \end{bmatrix}$$

Ha $\underline{\underline{Q}}$ nem tartalmaz n-elemű oszlop-bázist, akkor nem minden $\underline{z}$ vektor ($\underline{z}\in\mathbb{R}^n$) állítható elő az "integrálos" lineáris kombinációval

szükséges (és elégséges)

$$\boxed{\operatorname{rank}\left[\underline{\underline{B}} \,\vdots\, \underline{\underline{A}}\,\underline{\underline{B}} \,\vdots\, \ldots \,\vdots\, \underline{\underline{A}}^{n-1}\underline{\underline{B}}\right] = n}$$

$\boxed{e^{\underline{\underline{A}}t}}$ számítása
Scilab (MATLAB)
"gazdaságos" programozásával
(NEM csak a beépített expm( ))

```
// exp_At.sce
clc
A =[ 0  1;-2 -3]
t=1
exp_At=expm(A*t)

I=[ 1  0; 0  1];
Ak=I
SUM=Ak
for k=1:10
    k
    Ak=(A*t)*Ak;
    SUM = SUM + Ak/factorial(k)
end
```

```
// exp_At.sce
//===========================
clc
A =[ 0  1;-2 -3]
t=1
exp_At=expm(A*t)

I=[ 1  0; 0  1];
Ak=I
SUM=Ak
for k=1:10
   k
   Ak=(A*t)*Ak;
   SUM=SUM + Ak/factorial(k)
end
//===========================
```

```
t  =

   1.
-->exp_At=expm(A*t)
 exp_At  =

   0.6004236    0.2325442
 - 0.4650883  - 0.0972089
-->I=[ 1  0; 0  1];
-->Ak=I
 Ak  =

   1.    0.
   0.    1.
```

```
-->SUM=Ak
 SUM  =

   1.    0.
   0.    1.
-->for k=1:10
-->    k
-->    Ak=(A*t)*Ak;
-->    SUM = SUM +
Ak/factorial(k)
-->end
```

```
k  = 1.
 SUM  =
   1.    1.
 - 2.  - 2.

k  = 2.
 SUM  =
   0.  - 0.5
   1.    1.5

k  = 3.
 SUM  =
   1.            0.6666667
 - 1.3333333  - 1.

k  = 4.
 SUM  =
   0.4166667    0.0416667
 - 0.0833333    0.2916667

k  = 5.
 SUM  =
   0.6666667    0.3
 - 0.6        - 0.2333333

k  = 6.
 SUM  =
   0.5805556    0.2125
 - 0.425      - 0.0569444

k  = 7.
 SUM  =
   0.6055556    0.2376984
 - 0.4753968  - 0.1075397
```

```
k  = 8.
 SUM  =
   0.5992560    0.2313740
 - 0.4627480  - 0.0948661

k  = 9.
 SUM  =
   0.6006614    0.2327822
 - 0.4655644  - 0.0976852

k  = 10.
 SUM  =
   0.6003797    0.2325003
 - 0.4650006  - 0.0971211
```

```
exp_At  =
   0.6004236    0.2325442
 - 0.4650883  - 0.0972089
```

"pontos"

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