Konzultáció P0 kurzus 2024.06.17.

Sorok konvergenciája $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \overbrace{\sum_{n=1}^{k} a_n}^{= s_k}$

① $\lim_{n\to\infty} a_n \neq 0$, akkor divergens a sor!

$\sum_{n=1}^{\infty} 2^n = 2+4+8+\dots \to \infty$

(DE $\sum \frac{1}{n} = \infty$ $\lim_{n\to\infty} \frac{1}{n} = 0$)

② geom. sor $\sum_{n=0}^{\infty} a\cdot q^n = a\cdot 1 + a\cdot q + aq^2 \dots$

$$= \begin{cases} \text{ha } |q|<1 \text{ akkor konv. } \sum = a\cdot\frac{1}{1-q} \\ \text{ha } |q|\geq 1 \text{ akkor div.} \end{cases}$$

Pl: $\sum_{n=0}^{\infty} \frac{2^n + 3\cdot 5^{n-1}}{3^{2n-1}} = \sum_{n=0}^{\infty} \frac{2^n}{9^n\cdot 3^{-1}} + \frac{3\cdot 5^n\cdot 5^{-1}}{9^n\cdot 3^{-1}} =$

$\sum_{n=0}^{\infty} \overset{a}{3}\cdot\left(\frac{2}{9}\right)^n = 3\cdot\frac{1}{1-\frac{2}{9}} = 3\cdot\frac{9}{7} = \frac{27}{7}$

$0<q<1$

$$\sum_{n=0}^{\infty} \frac{3 \cdot 5^{n} \cdot 5^{-1}}{9^{n} \cdot 3^{-1}} = \sum_{n=0}^{\infty} \frac{3 \cdot \frac{1}{5}}{\frac{1}{3}} \left(\frac{5}{9}\right)^{n} = \frac{9}{5} \cdot \frac{1}{1-\frac{5}{9}} = \frac{9}{5} \cdot \frac{9}{4} = \frac{81}{20}$$

$\frac{9}{5} = a$ $\quad$ $0 < q < 1$

③ poz. és negatív tagok: Pl: $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} = -\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \dots$

Ha Leibniz: KONVERGENS

- alternál ($(-1)^n \cdot$ pozitív $= a_n$)
- $\lim_{n\to\infty} a_n = 0$ iff: $\lim_{n\to\infty} \frac{(-1)^n}{2n} = \frac{\pm 1}{\infty} = 0$ $\quad$ $2n \to \infty$
- $|a_{n+1}| < |a_n|$ mon. csökk.

$$\left|\frac{(-1)^{n+1}}{2(n+1)}\right| \overset{?}{<} \left|\frac{(-1)^n}{2n}\right|$$

$$\frac{1}{2n+2} \overset{?}{<} \frac{1}{2n}$$

$2n+2 > 2n$ ok

$2 > 0$

$\sum_{n=1}^{\infty} a_n$ vegyesen tartalmaz $\pm$ elemeket $\Longrightarrow$

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ vizsgáljuk

④ $a_n \geq 0$ pozitív tagú sor

Kritérium: Ⓘ – majoráns: Ha van $\sum b_n$ úgy hogy $0 \leq a_n \leq b_n$ és $\sum b_n$ konv $\Rightarrow$
$\Rightarrow \sum a_n$ konv.

Példa $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{konv. ha } p > 1 \\ \text{div. ha } p \leq 1 \end{cases}$

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^5+n}}}_{\text{ez is konv.}} \qquad a_n = \frac{1}{\sqrt[4]{n^5+n}} > 0 \qquad \overset{a_n}{\overset{\shortparallel}{\frac{1}{\sqrt[4]{n^5+n}}}} < \underbrace{\overset{b_n}{\frac{1}{\sqrt[4]{n^5}}} = \frac{1}{n^{5/4}}}_{\text{konv. majoráns}} \quad \frac{5}{4} > 1$$

$\sum_{n=1}^{\infty} \frac{1}{n^{5/4}} \to$ konv.

Ⓘⓘ – minoráns:

Ha $\sum b_n$ poz. tagú és divergens és $0 \leq b_n \leq a_n \Rightarrow \sum_{n=1}^{\infty} a_n$ divergens

Példa $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4-n-3}} \quad \frac{1}{\sqrt[5]{n^4-n-3}} > \frac{1}{\sqrt[5]{n^4}} = \frac{1}{n^{\frac{4}{5}}} \quad \frac{4}{5} < 1 \to \sum \frac{1}{n^{4/5}}$ div

divergens

III gyök.

$\sum_{n=1}^{\infty} a_n \to$ Ha $\lim_{n\to\infty} \sqrt[n]{a_n} = H < 1$ akkor konv

$H > 1$ akkor div.

$H = 1$ nem tudjuk

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} \to \lim_{n\to\infty} \frac{1}{\sqrt[n]{n \cdot 2^n}} = \lim_{n\to\infty} \frac{1}{\underbrace{\sqrt[n]{n}}_{\to 1} \cdot \underbrace{\sqrt[n]{2^n}}_{=2}} = \frac{1}{2} < 1$$

ezért $\sum \frac{1}{n \cdot 2^n}$ is konvergens!

IV hányados

$\sum_{n=1}^{\infty} a_n \to$ Ha $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = H$ <1 konv. >1 div $=1$ nem tudjuk

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} \to \lim_{n\to\infty} \frac{\frac{1}{(n+1)\cdot 2^{n+1}}}{\frac{1}{n \cdot 2^n}} = \lim_{n\to\infty} \frac{n \cdot \cancel{2^n}}{(n+1)\cancel{2^{n+1}}} = \lim_{n\to\infty} \underbrace{\frac{n}{n+1}}_{\to 1} \cdot \frac{1}{2} = \frac{1}{2}$$

$\sum a_n \to \sum |a_n| \to$ konv. $\Rightarrow \sum a_n$ is az
van $a_n < 0$ (abszolút konv.)

$\sum |a_n|$ divergens és $\sum a_n$ konvergens

$\Rightarrow \sum a_n$ feltételesen konvergens

$$\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}}_{\text{konv.}} \xrightarrow{|a_n|} \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2n} \right| = \underbrace{\sum_{n=1}^{\infty} \frac{1}{2n}}_{\text{div.}} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} \dots$$

Gyök: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{2n}} = \lim_{n\to\infty} \frac{1}{\underbrace{\sqrt[n]{2}}_{\to 1} \cdot \underbrace{\sqrt[n]{n}}_{\to 1}} = 1$? nem tudjuk

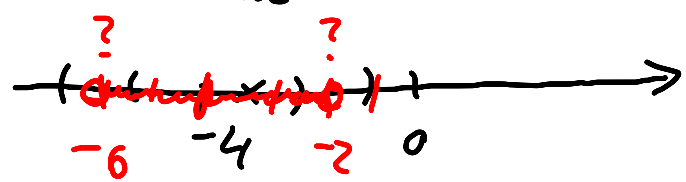
$\Rightarrow$ minoráns $\qquad \frac{1}{2n} = \frac{1}{2} \cdot \frac{1}{n}$

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{n} = \infty \text{ div.} \Rightarrow \text{divergens}$$

Hatványsor $\sum_{n=1}^{\infty} a_n \cdot (x-x_0)^n$ milyen x-re lesz konvergens?

Pl: $\sum_{n=1}^{\infty} \frac{(x+4)^n}{n\cdot 2^n} \to \sum_{n=1}^{\infty} \underbrace{\frac{1}{n\cdot 2^n}}_{=a_n} \cdot (x-\underbrace{(-4)}_{x_0})^n$

Gyökkrit



$$\lim_{n\to\infty} \sqrt[n]{\left|a_n (x-x_0)^n\right|} =$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{1}{n\cdot 2^n}\cdot |x+4|^n} = |x+4| \underbrace{\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}\cdot 2}}_{H} < 1$$

$$\left.|x+4| < \frac{1}{H}\right\}\ R \text{ konv. sug.}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}\cdot 2} = \frac{1}{2} \Rightarrow |x+4| < 2$$

ha $x \in (-6,-2)$ akkor konv.

Ha $x=-6$ $\sum_{n=1}^{\infty} \frac{1}{n\cdot 2^n} \cdot (\underbrace{-6+4}_{-2})^n = \sum_{n=1}^{\infty} \frac{-2^n}{n\cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1+\frac{1}{2}-\frac{1}{3} \ldots$

- alternál, $- \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$; - mon. cs. $|a_{n+1}| < |a_n|$

$\frac{1}{n+1} < \frac{1}{n}$ OK

$\Rightarrow$ konv. sor, mert Leibniz

Ha $x=-2$ $\sum_{n=1}^{\infty} \frac{1}{n\cdot 2^n} \cdot (\underbrace{-2+4}_{2})^n = \sum_{n=1}^{\infty} \frac{1}{n^{p}}$ $p=1$ $\Rightarrow$ divergens

$\rightarrow$ ha $x \in [-6,-2)$, akkor a hatványsor KONVERGENS

Kétváltozós függvény gradiense

$$f(x,y) = \frac{x^2+2y}{x\cdot e^y} \longrightarrow \begin{array}{l} f'_x = ? \\ f'_y = ? \end{array}$$

$$f'_x(x,y) = \left(\frac{x^2+2y}{x\cdot e^y}\right)'_x = \frac{2x\cdot x\cdot e^y-(x^2+2y)\cdot e^y}{(x\cdot e^y)^2} =$$

$$= \frac{2x^2\cdot\cancel{e^y}-x^2\cdot\cancel{e^y}-2y\cdot\cancel{e^y}}{x^2\cdot(e^y)^{\cancel{2}}} = \frac{x^2-2y}{x^2\cdot e^y}$$

$$f'_y(x,y) = \left(\frac{x^2+2y}{x\cdot e^y}\right)'_y = \frac{2\cdot x\cdot e^y-(x^2+2y)\cdot x\cdot e^y}{(x\cdot e^y)^2} =$$

$$= \frac{2x\cdot\cancel{e^y}-x^3\cdot\cancel{e^y}-2xy\cancel{e^y}}{x^2(e^y)^{\cancel{2}}} = \frac{2\cancel{x}-x^{\cancel{3}\,2}-2\cancel{x}y}{x^{\cancel{2}}\cdot e^y} = \frac{2-x^2-2y}{x\cdot e^y}$$

$P(-2,0) \Rightarrow$ itt a gradiens

$$f'_x(-2,0) = \left(\frac{x^2-2y}{x^2\cdot e^y}\right)\Bigg|_{(-2,0)} = \frac{(-2)^2-2\cdot 0}{(-2)^2\cdot \underbrace{e^0}_{=1}} = \frac{4}{4\cdot 1} = 1$$

$$f'_y(-2,0) = \left(\frac{2-x^2-2y}{x\cdot e^y}\right)\Bigg|_{(-2,0)} = \frac{2-(-2)^2-2\cdot 0}{(-2)\cdot \underbrace{e^0}_{=1}} = \frac{2-4}{-2} = 1$$

$$\text{grad } f(-2,0) = (1,1)$$

Érintősík a $(-2,0)$ pontban

$$z = \underbrace{f'_x(P)}_{=1}\cdot(x-\underbrace{x_0}_{-2}) + \underbrace{f'_y(P)}_{=1}(y-\underbrace{y_0}_{=0}) + \underbrace{f(x_0,y_0)}$$

$$f(-2,0) = \left(\frac{x^2+2y}{x\cdot e^y}\right)\Bigg|_{(-2,0)} = \frac{(-2)^2+2\cdot 0}{(-2)\cdot e^0} = \frac{4}{-2} = -2$$

Érintősík

$$z = 1\cdot(x-(-2)) + 1\cdot(y-0) + -2$$

$$z = x + \cancel{2} + y - \cancel{2}$$

$$z = x+y$$

$$0 = x+y-z$$

Mi lesz f iránymenti deriváltja az $\alpha = 120°$ irányban a $P(-2,0)$ pontban

$$f'_\alpha(-2,0) = \left\langle \overset{(1,1)}{\underline{\operatorname{grad}}(-2,0)}, \left(\overset{-\frac{1}{2}}{\cos 120°}, \overset{\frac{\sqrt{3}}{2}}{\sin 120°}\right)\right\rangle$$

$$= \left\langle (1,1), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right\rangle = -\frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-1}{2}$$

$\underline{v} = (-3, 4)$ irányban $P(-2, 0)$ pontban iránymenti

$$f'_{\underline{v}}(-2,0) = \left\langle \underline{\underline{\mathrm{grad}}}(-2,1) ; \frac{\underline{v}}{|\underline{v}|} \right\rangle =$$

$$|\underline{v}| = \sqrt{(-3)^2 + (4)^2} = \sqrt{25} = 5$$

$$= \left\langle (1, 1) ; \left(-\frac{3}{5}, \frac{4}{5}\right) \right\rangle =$$

$$= -\frac{3}{5} + \frac{4}{5} = \frac{1}{5}$$