Analytic geometry

1.1 Vectors in the Euclidean space

Definition 1.1 *The representing point pairs of a translation called vector.*

Definition 1.2 *The sum of two vectors is a vector, that we get by the parallelogram rule.*

Definition 1.3 *Let $t > 0$, $t \in \mathbb{R}$, then the multiplication of $\overrightarrow{PP'}$ by t is $\overrightarrow{PP''}$ if $\left|\overrightarrow{PP''}\right| = t\left|\overrightarrow{PP'}\right|$ and $(P'PP'')$ is not true. If $t = 0$, then $\overrightarrow{PP''} = \underline{0}$. If $t < 0$, then $\overrightarrow{PP''}$ is such that $\left|\overrightarrow{PP''}\right| = -t\left|\overrightarrow{PP'}\right|$ and $(P'PP'')$ is ture.*

Lemma 1.4 *The vectors of the space with the defined addition and scalar multiplication form a vector space.*

Definition 1.5 *Let $\underline{a}$ and $\underline{b}$ be vectors in the space, then $\langle \underline{a}, \underline{b} \rangle = |\underline{a}||\underline{b}|\cos(\gamma)$ called the dot product.*

Remark 1.6 $\langle \underline{a}, \underline{b} \rangle = 0 \Leftrightarrow a \perp b$, $\langle \underline{a}, \underline{b} \rangle > 0 \Leftrightarrow \gamma < \frac{\pi}{2}$, $\langle \underline{a}, \underline{b} \rangle < 0 \Leftrightarrow \gamma > \frac{\pi}{2}$.

Theorem 1.7 *Dot product is symmetric, positive definite, bilinear function.*

Lemma 1.8 *Let $\underline{a}$ and $\underline{v}$ be vectors. Then $\underline{v} = \underline{v}_\perp + \underline{v}_\parallel$, where $\underline{a} \parallel \underline{v}_\parallel$, $\underline{a} \perp \underline{v}_\perp$ and $\underline{v}_\parallel = \dfrac{\langle \underline{a}, \underline{v} \rangle}{\langle \underline{a}, \underline{a} \rangle}\underline{a}$, $\underline{v}_\perp = \underline{v} - \underline{v}_\parallel$.*

Proof. Let α be the angle between $\underline{a}$ and $\underline{v}$. Then $\dfrac{|\underline{v}_\parallel|}{|\underline{v}|} = \cos\alpha \Rightarrow |\underline{v}_\parallel| = |\underline{v}|\cos\alpha$, but $\langle \underline{a}, \underline{v} \rangle = |\underline{a}||\underline{v}|\cos\alpha$, therefore $|\underline{v}_\parallel| = \dfrac{\langle \underline{a}, \underline{v} \rangle}{|\underline{a}|}$. Now, we need a unit vector in direction of $\underline{a}$ to obtain $\underline{v}_\parallel$: $\underline{a}_u = \dfrac{\underline{a}}{|\underline{a}|}$. So that $\underline{v}_\parallel = |\underline{v}_\parallel|\underline{a}_u = \dfrac{\langle \underline{a}, \underline{v} \rangle}{|\underline{a}|^2}\underline{a} = \dfrac{\langle \underline{a}, \underline{v} \rangle}{\langle \underline{a}, \underline{a} \rangle}\underline{a}$. ∎

Definition 1.9 *Let $\underline{a}$ and $\underline{b}$ be vectors, then $\underline{a} \times \underline{b}$ is the cross product of $\underline{a}$ and $\underline{b}$, where $|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin\gamma$, $\underline{a} \times \underline{b}$ is orthogonal both $\underline{a}$ and $\underline{b}$ and the direction of $\underline{a} \times \underline{b}$ is given by the right-hand rule.*

Theorem 1.10 *Cross product is antisymmetric bilinear function.*

Remark 1.11 *$\underline{a} \times \underline{b} = \underline{0} \Leftrightarrow \underline{a} \parallel \underline{b}$, and $|\underline{a} \times \underline{b}|$ is equal to the area of the spanned parallelogram.*

Definition 1.12 *Let $\underline{a}$, $\underline{b}$ and $\underline{c}$ be vectors in the space. Then the triple product of them is $\underline{a} \cdot \underline{b} \cdot \underline{c} = \langle \underline{a} \times \underline{b}, \underline{c} \rangle$.*

Theorem 1.13 *Triple product is invariant under a circular shift: $\underline{a} \cdot \underline{b} \cdot \underline{c} = \langle \underline{a} \times \underline{b}, \underline{c} \rangle = \langle \underline{b} \times \underline{c}, \underline{a} \rangle = \langle \underline{c} \times \underline{a}, \underline{b} \rangle$.*

Proof. This is true because of the geometric meaning. $\underline{a} \cdot \underline{b} \cdot \underline{c}$ is the (signed) volume of the spanned parallelepiped. The area, spanned by $\underline{a}$ and $\underline{b}$ is exactly $|\underline{a} \times \underline{b}|$ and $\frac{h}{|\underline{c}|} = \cos(\underline{a} \times \underline{b}, \underline{c}) \Rightarrow h = |\underline{c}| \cos\gamma$. Then the volume of the spanned parallelepiped is $V = |\underline{a} \times \underline{b}||\underline{c}| \cos\gamma = \langle \underline{a} \times \underline{b}, \underline{c} \rangle = \underline{a} \cdot \underline{b} \cdot \underline{c}$. ■

Theorem 1.14 (Lagrange's formula) $(\underline{a} \times \underline{b}) \times \underline{c} = \langle \underline{a}, \underline{c} \rangle \underline{b} - \langle \underline{b}, \underline{c} \rangle \underline{a}$

Proof. $(\underline{a} \times \underline{b}) \times \underline{c} \perp \underline{a} \times \underline{b} \Rightarrow (\underline{a} \times \underline{b}) \times \underline{c}$ is in the plane of $\underline{a}$ and $\underline{b} \Rightarrow (\underline{a} \times \underline{b}) \times \underline{c} = \alpha\underline{a} + \beta\underline{b}$
Now, we consider the dot product of both sides with $\underline{b} \times \underline{c}$:
Left side: $\langle (\underline{a} \times \underline{b}) \times \underline{c}, \underline{b} \times \underline{c} \rangle = \langle \underline{c} \times (\underline{b} \times \underline{c}), \underline{a} \times \underline{b} \rangle$. Now, if $\underline{v} := \underline{c} \times (\underline{b} \times \underline{c})$, then $|\underline{v}| = |\underline{c}||\underline{b} \times \underline{c}|$, since $\underline{c} \perp \underline{b} \times \underline{c}$. But than $|\underline{v}| = |\underline{c}|^2 |\underline{b}| \sin\gamma$, where γ is the angle of $\underline{b}$ and $\underline{c}$. If we decompose $\underline{b}$ into parallel and orthogonal elements respected to $\underline{c}$, then $|\underline{b}_\perp| = |\underline{b}| \sin\gamma$, therefore $\frac{|\underline{v}|}{|\underline{c}|^2} = |\underline{b}\perp|$. It can be seen, that $\underline{v} \parallel \underline{b}_\perp \Rightarrow \underline{v} = |\underline{c}|^2 \underline{b}_\perp = |\underline{c}|^2(\underline{b} - \underline{b}_\parallel) = |\underline{c}|^2 \left(\underline{b} - \frac{\langle \underline{b}, \underline{c} \rangle}{|\underline{c}|^2} \underline{c} \right) = |\underline{c}|^2 \underline{b} - \langle \underline{b}, \underline{c} \rangle \underline{c}$. Finally, $\langle \underline{c} \times (\underline{b} \times \underline{c}), \underline{a} \times \underline{b} \rangle = \langle |\underline{c}|^2 \underline{b} - \langle \underline{b}, \underline{c} \rangle \underline{c}, \underline{a} \times \underline{b} \rangle = 0 - \langle \underline{b}, \underline{c} \rangle \langle \underline{c}, \underline{a} \times \underline{b} \rangle = - \langle \underline{b}, \underline{c} \rangle (\underline{a} \cdot \underline{b} \cdot \underline{c})$.
Right side: $\alpha \langle \underline{a}, \underline{b} \times \underline{c} \rangle = \alpha(\underline{a} \cdot \underline{b} \cdot \underline{c})$ and $\underline{b} \perp \underline{b} \times \underline{c} \Rightarrow \beta \langle \underline{b}, \underline{b} \times \underline{c} \rangle = 0$
Finally, we get that $\alpha = - \langle \underline{b}, \underline{c} \rangle$. Similarly, $\beta = \langle \underline{a}, \underline{c} \rangle$. ■

Theorem 1.15 (Jacobi's formula) $(\underline{a} \times \underline{b}) \times \underline{c} + (\underline{b} \times \underline{c}) \times \underline{a} + (\underline{c} \times \underline{a}) \times \underline{b} = 0$

Proof. $(\underline{b} \times \underline{c}) \times \underline{a} = -\underline{a} \times (\underline{b} \times \underline{c})$ and $(\underline{c} \times \underline{a}) \times \underline{b} = -(\underline{a} \times \underline{c}) \times \underline{b}$ due to the antisymmetry.
$(\underline{a} \times \underline{b}) \times \underline{c} - \underline{a} \times (\underline{b} \times \underline{c}) = \langle \underline{a}, \underline{c} \rangle \underline{b} - \langle \underline{b}, \underline{c} \rangle \underline{a} + \langle \underline{a}, \underline{b} \rangle \underline{c} - \langle \underline{a}, \underline{c} \rangle \underline{b} = \langle \underline{a}, \underline{b} \rangle \underline{c} - \langle \underline{b}, \underline{c} \rangle \underline{a} = (\underline{a} \times \underline{c}) \times \underline{b}$
■

Lemma 1.16 $\langle \underline{a} \times \underline{b}, \underline{a} \times \underline{b} \rangle = \langle \underline{a}, \underline{a} \rangle \langle \underline{b}, \underline{b} \rangle - \langle \underline{a}, \underline{b} \rangle^2$

Proof. $\langle \underline{a} \times \underline{b}, \underline{a} \times \underline{b} \rangle = |\underline{a} \times \underline{b}|^2 = |\underline{a}|^2 |\underline{b}|^2 \sin^2\gamma = |\underline{a}|^2 |\underline{b}|^2 (1 - \cos^2\gamma) = |\underline{a}|^2 |\underline{b}|^2 - |\underline{a}|^2 |\underline{b}|^2 \cos^2\gamma = \langle \underline{a}, \underline{a} \rangle \langle \underline{b}, \underline{b} \rangle - \langle \underline{a}, \underline{b} \rangle^2$ ■

Theorem 1.17

$$\langle \underline{a}, \underline{b} \rangle = \underline{a}^T \underline{b}, \qquad \underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \qquad \underline{a} \cdot \underline{b} \cdot \underline{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Point: (x, y, z)
Line: Let P_0 be a point on the line l and $\underline{v}$ its direction. Then $P \in l \Leftrightarrow \exists t \in \mathbb{R} : P = P_0 + t\underline{v}$ $\Rightarrow x = x_0 + tv_1,\ y = y_0 + tv_2,\ z = z_0 + tv_3$ or $\dfrac{x - x_0}{v_1} = \dfrac{y - y_0}{v_2} = \dfrac{z - z_0}{v_3}$ if $v_i \neq 0$.
Plane: Let P_0 be a point on the plane α and $\underline{n}(A, B, C)$ be orthogonal to the plane. Then $P \in \alpha \Leftrightarrow \overrightarrow{P_0P} \perp \underline{n} \Leftrightarrow \left\langle \overrightarrow{P_0P}, \underline{n} \right\rangle = 0 \Leftrightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = Ax + By + Cz + D = 0$, where $D = -(Ax_0 + By_0 + Cz_0)$.

1.2 Transformation of $\mathbf{E}^3$ with one fix point (origin)

1. Reflection in a plane through the origin

   $\underline{p}' = \underline{p} + \overrightarrow{PP'} = \underline{p} + 2\overrightarrow{PT} = \underline{p} - 2\underline{p}_{\|} = \underline{p} - 2\dfrac{\langle \underline{p}, \underline{n} \rangle}{\langle \underline{n}, \underline{n} \rangle}\underline{n}$,
   where $\underline{n} = (A, B, C)^T$ such that $|\underline{n}| = 1$.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - 2\begin{bmatrix} Ax & By & Cz \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} =$$

$$\begin{bmatrix} 1 - 2A^2 & -2AB & -2AC \\ -2AB & 1 - 2B^2 & -2BC \\ -2AC & -2BC & 1 - 2C^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_\alpha \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

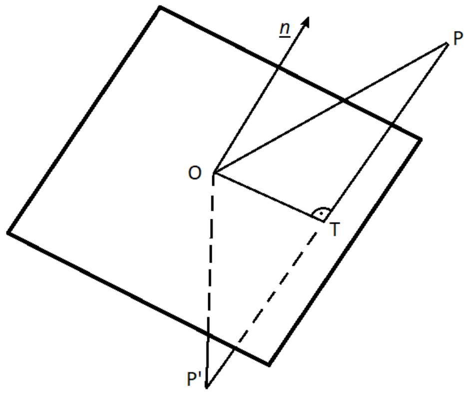


Figure 1.1: Reflection in plane

2. Projection to a plane through the origin

   $\underline{p}' = \underline{p} - \underline{p}_{\|} \Rightarrow$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 - A^2 & -AB & -AC \\ -AB & 1 - B^2 & -BC \\ -AC & -BC & 1 - C^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = P_\alpha \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

   where $\underline{n} = (A, B, C)^T$ such that $|\underline{n}| = 1$.

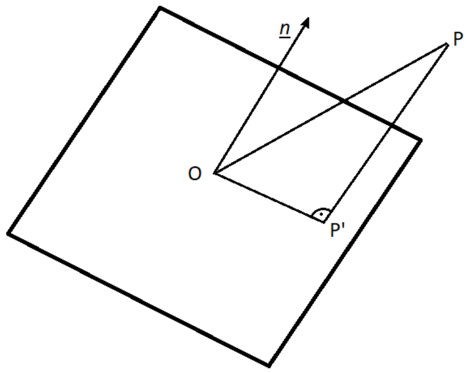


Figure 1.2: Projection in plane

3. Reflection in a line through the origin
   First, we reflect P in the plane, orthogonal to the line l to obtain P^*, then we reflect P^* in the origin to obtain P'. Let $\underline{v}_l = \underline{n} = (A, B, C)^T$, such that $|\underline{v}_e| = 1$.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = -\underline{\underline{I}} \cdot \begin{bmatrix} 1-2A^2 & -2AB & -2AC \\ -2AB & 1-2B^2 & -2BC \\ -2AC & -2BC & 1-2C^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$= \begin{bmatrix} 2A^2-1 & 2AB & 2AC \\ 2AB & 2B^2-1 & 2BC \\ 2AC & 2BC & 2C^2-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_l \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

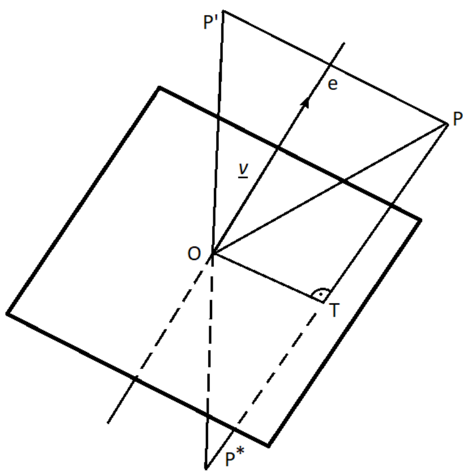


Figure 1.3: Reflection in line

4. Projection to a line through the origin
   $\overrightarrow{OP} = \underline{p} = \overrightarrow{OP'} + \overrightarrow{OP^*} \Rightarrow \underline{p}' = \underline{p} - \overrightarrow{OP^*}$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underline{\underline{I}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1-A^2 & -AB & -AC \\ -AB & 1-B^2 & -BC \\ -AC & -BC & 1-C^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$= \begin{bmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = P_e \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{v}_e \underline{v}_e^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

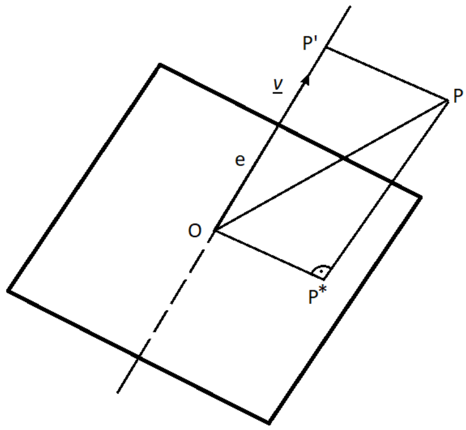


Figure 1.4: Projection in line

5. Left cross product with a fix vector

$$\underline{p}' = \underline{n} \times \underline{p} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A & B & C \\ x & y & z \end{vmatrix} = \begin{bmatrix} Bz - Cy \\ Cx - Az \\ Ay - Bx \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -C & B \\ C & 0 & -A \\ -B & A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = C_{\underline{n}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

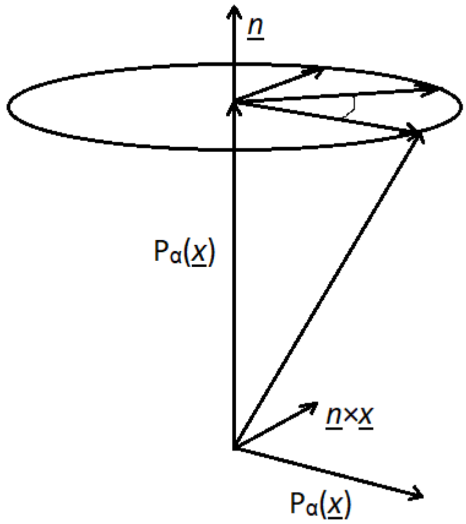


Figure 1.5: Rotation around a line

6. Rotation around a line, through the origin
   $\underline{p}' = P_l(\underline{p}) + \cos\phi P_\alpha(\underline{p}) + \sin\phi C_{\underline{n}}(\underline{p}) \Rightarrow$

$$\begin{bmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{bmatrix} + \sin\phi \begin{bmatrix} 0 & -C & B \\ C & 0 & -A \\ -B & A & 0 \end{bmatrix} +$$

$$+ \cos\phi \begin{bmatrix} 1-A^2 & -AB & -AC \\ -AB & 1-B^2 & -BC \\ -AC & -BC & 1-C^2 \end{bmatrix} = R_{\underline{n}}^{\phi}$$

Definition 1.18 *The homogeneous coordinates of a spatial point P is the equivalence class of the assigned point quartet.*

$$P(\underline{p}) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \hookrightarrow \tilde{\underline{p}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \sim \left\{ \alpha \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \setminus \{0\} \right\}.$$

Q is an ideal point, if $Q \sim [x, y, z, 0]^T$, but $\nexists R \sim [0, 0, 0, 0]^T$.

7. Translation with $\underline{t}$

$$\underline{p}' = \underline{p} + \underline{t} = \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = T_{\underline{t}} \sim \left\{ \alpha \cdot \begin{bmatrix} \underline{\underline{I}} & \underline{t} \\ \underline{0}^T & 1 \end{bmatrix}, \alpha \in \mathbb{R} \setminus \{0\} \right\}$$

8. Linear transformations

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} \Rightarrow \begin{bmatrix} \underline{x}' \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{\underline{A}} & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix}$$

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{b} \Rightarrow \begin{bmatrix} \underline{x}' \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{\underline{A}} & \underline{b} \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix}$$

9. Point on a line

   Let $X(\underline{x})$ and $Y(\underline{y})$ be two points on a line l. Then $U(\underline{u}) \in l \Leftrightarrow \exists \alpha \underline{u} = \underline{y} + \alpha(\underline{x} - \underline{y}) = \alpha \underline{x} + (1 - \alpha)\underline{y})$. Now let α and β be real numbers, then:

$$\alpha \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix} + \beta \begin{bmatrix} \underline{y} \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \underline{x} + \beta \underline{y} \\ \alpha + \beta \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\alpha+\beta} \underline{x} + \frac{\beta}{\alpha+\beta} \underline{y} \\ 1 \end{bmatrix}$$

10. Rotation around a line

    (a) Translation of a point of the line to the origin: $\begin{bmatrix} \underline{\underline{I}} & -\underline{t} \\ \underline{0}^T & 1 \end{bmatrix}$

    (b) Rotation around a line through the origin: $\begin{bmatrix} \underline{\underline{R}}_{\underline{n}}^{\alpha} & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix}$

    (c) Inverse translation: $\begin{bmatrix} \underline{\underline{I}} & \underline{t} \\ \underline{0}^T & 1 \end{bmatrix}$

$$\begin{bmatrix} \underline{\underline{I}} & \underline{t} \\ \underline{0}^T & 1 \end{bmatrix} \cdot \begin{bmatrix} \underline{\underline{R}}_{\underline{n}}^{\alpha} & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix} \cdot \begin{bmatrix} \underline{\underline{I}} & -\underline{t} \\ \underline{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \underline{\underline{I}} & \underline{t} \\ \underline{0}^T & 1 \end{bmatrix} \cdot \begin{bmatrix} \underline{\underline{R}}_{\underline{n}}^{\alpha} & -\underline{\underline{R}}_{\underline{n}}^{\alpha} \underline{t} \\ \underline{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \underline{\underline{R}}_{\underline{n}}^{\alpha} & \underline{t} - \underline{\underline{R}}_{\underline{n}}^{\alpha} \underline{t} \\ \underline{0}^T & 1 \end{bmatrix}$$

Definition 1.19 *If, X, Y and Z are collinear points such that $\underset{\sim}{\tilde{z}} = \alpha\underset{\sim}{\tilde{x}} + \beta\underset{\sim}{\tilde{y}}$, then $(X,Y,Z) := \frac{\beta}{\alpha}$ is called division ratio. If, X, Y, Z and W are collinear points such that $\underline{\tilde{z}} = \alpha_1\underline{\tilde{x}} + \beta_1\underline{\tilde{y}}$ and $\underline{\tilde{w}} = \alpha_2\underline{\tilde{x}} + \beta_2\underline{\tilde{y}}$ then $(X,Y,Z,W) := \frac{\beta_1}{\alpha_1} : \frac{\beta_2}{\alpha_2} = \frac{(X,Y,Z)}{(X,Y,W)}$ is called cross ratio.*

Lemma 1.20 *The linear transformation $\tilde{A} = \begin{bmatrix} \underline{\underline{A}} & t \\ \underline{u}^T & 1 \end{bmatrix}$ preserves cross ration and preserves division ratio if and only if $\underline{u} = \underline{0}$.*

Proof. $\underline{\tilde{z}} = \alpha_1\underline{\tilde{x}} + \beta_1\underline{\tilde{y}} \Rightarrow \widetilde{A\underline{\tilde{z}}} = \alpha_1\tilde{A}\underline{\tilde{x}} + \beta_1\tilde{A}\underline{\tilde{y}} = \alpha_1 \begin{bmatrix} \underline{\underline{A}} & t \\ \underline{u}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix} + \beta_1 \begin{bmatrix} \underline{\underline{A}} & t \\ \underline{u}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{y} \\ 1 \end{bmatrix} =$

$$\alpha_1 \begin{bmatrix} \underline{\underline{A}}\,\underline{x} + t \\ \underline{u}^T\underline{x} + 1 \end{bmatrix} + \beta_1 \begin{bmatrix} \underline{\underline{A}}\,\underline{y} + t \\ \underline{u}^T\underline{y} + 1 \end{bmatrix} = \frac{\alpha_1}{\underline{u}^T\underline{x}+1}\widetilde{A(\underline{\tilde{x}})} + \frac{\beta_1}{\underline{u}^T\underline{y}+1}\widetilde{A(\underline{\tilde{y}})} \Rightarrow$$

$$(\widetilde{A(\underline{\tilde{x}})}, \widetilde{A(\underline{\tilde{y}})}, \widetilde{A(\underline{\tilde{z}})}) = \frac{\frac{\beta_1}{\underline{u}^T\underline{y}+1}}{\frac{\alpha_1}{\underline{u}^T\underline{x}+1}} = \frac{\beta_1}{\alpha_1} \Leftrightarrow \underline{u}^T\underline{x} = \underline{u}^T\underline{y} \Leftrightarrow \underline{u} = \underline{0} \text{ and}$$

$$(\widetilde{A(\underline{\tilde{x}})}, \widetilde{A(\underline{\tilde{y}})}, \widetilde{A(\underline{\tilde{z}})}, \widetilde{A(\underline{\tilde{w}})}) = \frac{\frac{\beta_1}{\underline{u}^T\underline{y}+1}}{\frac{\alpha_1}{\underline{u}^T\underline{x}+1}} : \frac{\frac{\beta_2}{\underline{u}^T\underline{y}+1}}{\frac{\alpha_2}{\underline{u}^T\underline{x}+1}} = \frac{\beta_1}{\alpha_1} : \frac{\beta_2}{\alpha_2} = (X,Y,Z,W). \quad \blacksquare$$

Definition 1.21 *A collineation is a spatial transformation, which is a bijection respected to points and lines, and preserves incidence.*

Theorem 1.22 *Any collineation can be represented (in homogeneous coordinates) by the equivalent class of a regular linear transformation given in $\tilde{A} = \begin{bmatrix} \underline{\underline{A}} & t \\ \underline{u}^T & 1 \end{bmatrix}$ form.*

Definition 1.23 *A collineation is called affinity if $\underline{u} = \underline{0}$. Similarity is an affinity, which preserves the ratio of segments. Isometry is a similarity, which preserves the length of the segments.*

Lemma 1.24 *A collineation is similarity, if $\underline{u} = \underline{0}$ and $\underline{\underline{A}}$ is orthogonal.*

Proof. $\lambda^2 \left\langle \underline{x} - \underline{y}, \underline{x} - \underline{y} \right\rangle = \left\langle \underline{x}' - \underline{y}', \underline{x}' - \underline{y}' \right\rangle = \left\langle \underline{\underline{A}}(\underline{x} - \underline{y}), \underline{\underline{A}}(\underline{x} - \underline{y}) \right\rangle = \left\langle (\underline{x} - \underline{y}), \underline{\underline{A}}^T\underline{\underline{A}}(\underline{x} - \underline{y}) \right\rangle$
But $\lambda^2 \left\langle \underline{x} - \underline{y}, \underline{x} - \underline{y} \right\rangle = \left\langle (\underline{x} - \underline{y}), \lambda^2\underline{\underline{E}}(\underline{x} - \underline{y}) \right\rangle \Rightarrow \underline{\underline{A}}^T\underline{\underline{A}} = \lambda^2\underline{\underline{E}}$. $\blacksquare$

Remark 1.25 *If $\lambda = 1$ then we have an isometry.*

1.3 Spherical geometry

Points: surface of the sphere
Lines: shortest path between two points on the surface of the sphere $\Rightarrow$ great arcs $\Rightarrow$ great circles
Problem: Betweeness is not appropriate

1.3.1 Spherical Order axioms

Axiom Os1 *If $(ACBD)$ then $(BCAD)$ and $(CBDA)$.*

Axiom Os2 *If A, B and D are collinear points then there exists at least one point C such that $(ABCAD)$.*

Axiom Os3 *If four points are situated on a line, there is no more than one 2-2 partition, such that they separate each other.*

Axiom Os4 (Pasch) *Let A, B and C be three points not lying on the same line and e and f be lines, not passing through any of the points A, B, C. Then there exist the points $E \in e$ and $F \in f$ such that either $(AECF)$ or $(BECF)$.*

Remark 1.26 *If f is the ideal line, we get the Euclidean geometry.*

1.3.2 Distance, angle and area

Elliptic geometry: Antipodal points are united: $A = A'$.
Spherical geometry: Every point on the surface of the sphere. $A \neq A'$.
Distance of points: Length of the arc: $r \cdot \alpha = \alpha$ if $r = 1$.
If $\underline{x} = (x, y, z)^T$, then $\underline{x} \sim \underline{\tilde{x}} = (x', y', 1)^T$ if $z \neq 0$.

$$\cos\alpha = \frac{\left\langle \underline{\tilde{a}}, \underline{\tilde{b}} \right\rangle}{\sqrt{\left\langle \underline{\tilde{a}}, \underline{\tilde{a}} \right\rangle \left\langle \underline{\tilde{b}}, \underline{\tilde{b}} \right\rangle}}$$

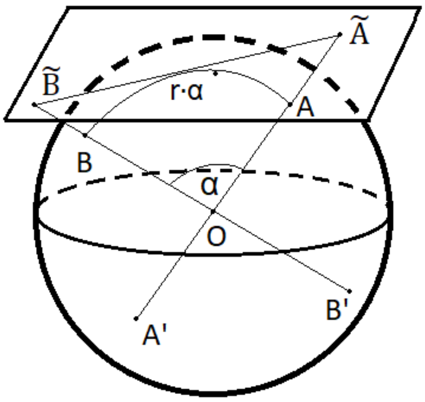


Figure 1.6: Spherical geometry

Lines: Great (half-)circles of the sphere.
Angel of lines: Angles of the orthogonal vectors of the determined planes:

$$\cos\alpha = \frac{\langle \underline{\tilde{n}_1}, \underline{\tilde{n}_2} \rangle}{\sqrt{\langle \underline{\tilde{n}_1}, \underline{\tilde{n}_1} \rangle \langle \underline{\tilde{n}_2}, \underline{\tilde{n}_2} \rangle}}$$

Definition 1.27 *Two lines on the sphere divide the surface into four spherical lune.*

Area of spherical lune:

$$A_{sphere} = 4\pi \Rightarrow A_{lune} = 2\alpha$$

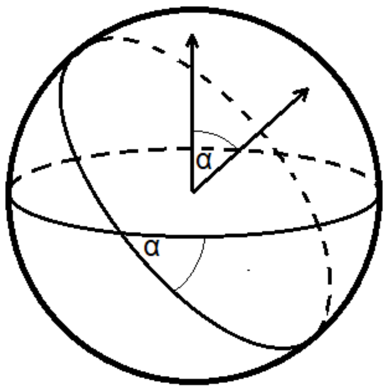


Figure 1.7: Angel of lines, spherical lune

Area of the spherical triangle:

We cover of the sphere with lunes:

$$2(2\alpha + 2\beta + 2\gamma) = 4\pi + 4A_{triangle} \Rightarrow$$

$$A_{triangle} = \alpha + \beta + \gamma - \pi$$

Now, let $\underline{a}$, $\underline{b}$ and $\underline{c}$ be vectors to A, B and C such that they form a right-hand system. Then $(\underline{c} \times \underline{a}, \underline{a} \times \underline{b})\angle = \pi - \alpha$, $(\underline{a} \times \underline{b}, \underline{b} \times \underline{c})\angle = \pi - \beta$, $(\underline{b} \times \underline{c}, \underline{c} \times \underline{a})\angle = \pi - \gamma$.

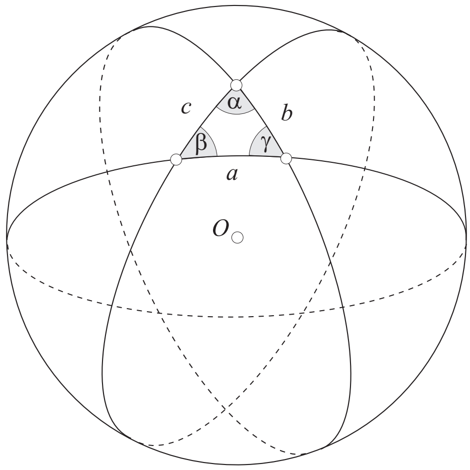


Figure 1.8: Spherical triangle

Theorem 1.28 (Spherical sine theorem) *Let a, b and c be the sides opposite to, and α, β and γ be the angles at the vertices A, B and C of a spherical triangle. Then* $\dfrac{\sin a}{\sin \alpha} = \dfrac{\sin b}{\sin \beta} = \dfrac{\sin c}{\sin \gamma}$.

Proof. According to the Lagrange formula and the definition of the triple product:

$$(\underline{a} \times \underline{b}) \times (\underline{b} \times \underline{c}) = \langle \underline{a}, \underline{b} \times \underline{c} \rangle \, \underline{b} - \langle \underline{b}, (\underline{b} \times \underline{c}) \rangle \, \underline{a} = (\underline{a} \cdot \underline{b} \cdot \underline{c})\underline{b},$$

since $\underline{b} \perp (\underline{b} \times \underline{c})$. This implies that $|(\underline{a} \times \underline{b}) \times (\underline{b} \times \underline{c})| = |\underline{a} \cdot \underline{b} \cdot \underline{c}|$, but $|(\underline{a} \times \underline{b}) \times (\underline{b} \times \underline{c})| = |\underline{a} \times \underline{b}| \cdot |\underline{b} \times \underline{c}| \sin(\pi - \beta) = \sin c \cdot \sin a \cdot \sin \beta$. Similarly, $|(\underline{c} \times \underline{a}) \times (\underline{c} \times \underline{b})| = |\underline{c} \cdot \underline{a} \cdot \underline{b}| = |\underline{a} \cdot \underline{b} \cdot \underline{c}|$ and $|(\underline{c} \times \underline{a}) \times (\underline{c} \times \underline{b})| = \sin b \cdot \sin c \cdot \sin \alpha$. Finally, we obtain that $\dfrac{\sin a}{\sin \alpha} = \dfrac{\sin b}{\sin \beta}$. The other equation can be proved similarly. ■

Theorem 1.29 (Spherical side cosine theorem) *Let a, b and c be the sides opposite to, and α, β and γ be the angles at the vertices A, B and C of a spherical triangle. Then* $\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos \gamma$.

Proof. On the one hand, $\langle (\underline{a} \times \underline{b}), (\underline{b} \times \underline{c}) \rangle = \langle (\underline{a} \times \underline{b}) \times \underline{b}, \underline{c} \rangle = \langle \langle \underline{a}, \underline{b} \rangle \, \underline{b} - \langle \underline{b}, \underline{b} \rangle \, \underline{a}, \underline{c} \rangle = \langle \underline{a}, \underline{b} \rangle \langle \underline{b}, \underline{c} \rangle - \langle \underline{a}, \underline{c} \rangle = \cos c \cdot \cos a - \cos b$. On the other hand, $\langle (\underline{a} \times \underline{b}), (\underline{b} \times \underline{c}) \rangle = |\underline{a} \times \underline{b}| \cdot |\underline{b} \times \underline{c}| \cos(\pi - \beta) = -\sin c \cdot \sin a \cdot \cos \beta \Rightarrow \cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos \beta$. Similarly, $\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos \gamma$ and $\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos \alpha$. ■

Definition 1.30 *Let $ABC\triangle$ be a spherical triangle. Then the $A' := \underline{b} \times \underline{c}$, $B' := \underline{c} \times \underline{a}$ and $C' := \underline{a} \times \underline{b}$ form the polar triangle of $ABC\triangle$ (The distances of the corresponding points are less then $\frac{\pi}{2}$).*

Theorem 1.31 *Let $ABC\triangle$ be a spherical triangle and $A'B'C'\triangle$ be its polar triangle. Then the polar triangle of $A'B'C'\triangle$ is $ABC\triangle$.*

Proof. We know, that the length of the arcs AB', AC', BA', BC', CA' and CB' are equal to $\frac{\pi}{2}$ and the length of the arcs AA', BB' and CC' is less then $\frac{\pi}{2}$, therefore the points A, B and C satisfy the conditions for A'', B'' and C''. ■

Theorem 1.32 *Let $ABC\triangle$ be a spherical triangle with angles α, β and γ, and $A'B'C'\triangle$ be its polar triangle with sides a', b' and c'. Then $a' + \alpha = b' + \beta = c' + \gamma = \pi$, if they are the corresponding sides and angles.*

Proof. $a' = (\underline{c} \times \underline{a}, \underline{a} \times \underline{b})\angle = \pi - \alpha$ ■

Theorem 1.33 (Spherical angle cosine theorem) *Let a, b and c be the sides opposite to, and α, β and γ be the angles at the vertices A, B and C of a spherical triangle. Then $\cos\gamma = -\cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta \cdot \cos c$.*

Proof. We apply the spherical side cosine theorem for the polar triangle $A'B'C'\triangle$: $\cos c' = \cos a' \cdot \cos b' + \sin a' \cdot \sin b' \cdot \cos\gamma' \Rightarrow \cos(\pi-\gamma) = \cos(\pi-\alpha)\cdot\cos(\pi-\beta) + \sin(\pi-\alpha)\cdot\sin(\pi-\beta)\cdot\cos(\pi-c)$. Now applying that $\sin(\pi-\phi) = \sin\phi$ and $\cos(\pi-\phi) = -\cos\phi$ we obtain that $-\cos\gamma = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta\cdot\cos c$. Similarly, $\cos\alpha = -\cos\beta\cdot\cos\gamma + \sin\beta\cdot\sin\gamma\cdot\cos a$ and $\cos\beta = -\cos\alpha\cdot\cos\gamma + \sin\alpha\cdot\sin\gamma\cdot\cos b$. ■

1.4 n-dimensional Euclidean geometry

Definition 1.34 *Let $(\mathbf{V}, \langle\cdot,\cdot\rangle)$ be a vector space with the usual dot product and $\mathbf{E}^n$ be a set. If the ordered point pairs of $\mathbf{E}^n$ can be assigned bijectively to the elements of the vectors space satisfying the following properties, then $\mathbf{E}^n$ is the n-dimensional (analytic) Euclidean space:*

- $\forall P, Q \in \mathbf{E}^n \exists! \underline{v} \in \mathbf{V} : \overrightarrow{PQ} = \underline{v}$
- $\forall P \in \mathbf{E}^n : \overrightarrow{PP} = \underline{0}$
- $\forall P \in \mathbf{E}^n, \underline{v} \in \mathbf{V} \exists! Q \in \mathbf{E}^n : \overrightarrow{PQ} = \underline{v}$
- $\forall P, Q, R \in \mathbf{E}^n : \overrightarrow{PR} + \overrightarrow{RQ} + \overrightarrow{QP} = \underline{0}$

Definition 1.35 *If $\mathbf{V}_k$ is a k-dimensional subspace of $\mathbf{V}$ and $\underline{x}^0 \in \mathbf{E}^n$, then the $\underline{X} = \underline{x}^0 + \mathbf{V}_k$ set is called a k-dimensional affine subspace of $\mathbf{E}^n$. If $k = 1$, then it is called line, if $k = n-1$, then it is called hyperplane.*

- $k = 1:\ \underline{x} = \underline{x}^0 + t \cdot \underline{v} \Rightarrow \frac{x_1 - x_1^0}{v_1} = \frac{x_2 - x_2^0}{v_2} = \cdots = \frac{x_n - x_n^0}{v_n}$, and $x_i = x_i^0$ if $v_i = 0$
- $k = n-1:\ \underline{x} = \underline{x}^0 + \sum_{i=1}^{n-1} \alpha_i \underline{v}_i \Rightarrow \{\underline{x} - \underline{x}^0, \underline{v}_1, \ldots, \underline{v}_{n-1}\}$ not linearly independent $\Rightarrow \exists! \underline{n} \neq \underline{0} : \langle \underline{n}, \underline{x} - \underline{x}^0 \rangle = 0 \Rightarrow \langle \underline{x}, \underline{n} \rangle = c$

1.5 Classification of quadratic surfaces

$$(\text{Q}): \quad \underline{x}^T \underline{\underline{A}}\,\underline{x} + 2\underline{b}^T \underline{x} + c = 0$$

is a quadratic form, where $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$, $\underline{b}, \underline{x} \in \mathbb{R}^n$, $c \in \mathbb{R}$ and $\underline{\underline{A}} = \underline{\underline{A}}^T$. By homogeneous coordinates $\underline{X} = \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix}$:

$$(\text{Q}^*): \quad \underline{X}^T \begin{bmatrix} \underline{\underline{A}} & \underline{b} \\ \underline{b}^T & c \end{bmatrix} \underline{X} = 0$$

Now, we change the coordinate system by translating the $\underline{\underline{B}}$ orthogonal system by $\underline{t}$. Then

$$(\text{Q}): \quad \underline{X}^T \begin{bmatrix} \underline{\underline{B}}^T \underline{\underline{A}}\,\underline{\underline{B}} & \underline{\underline{B}}^T (\underline{\underline{A}}\,\underline{t} + \underline{b}) \\ (\underline{t}^T \underline{\underline{A}} + \underline{b}^T)\underline{\underline{B}} & \underline{t}^t \underline{\underline{A}}\,\underline{t} + \underline{b}^T \underline{t} + \underline{t}^T \underline{b} + c \end{bmatrix} \underline{X} = 0$$

$\underline{\underline{B}}$ can be chosen such that $\underline{\underline{B}}^T \underline{\underline{A}}\,\underline{\underline{B}}$ is diagonal and λ_i are the diagonal elements $i = 1 \dots n$.
1. case $\forall i : \lambda_i \neq 0$ $\underline{\underline{A}}$ is invertible and $\underline{t} = -\underline{\underline{A}}^{-1} \underline{b}$.

$$(\text{Q}): \quad \underline{X}^T \begin{bmatrix} \Lambda & \underline{0} \\ \underline{0}^T & -\underline{b}^T \underline{\underline{A}}^{-1} \underline{b} + c \end{bmatrix} \underline{X} = 0,$$

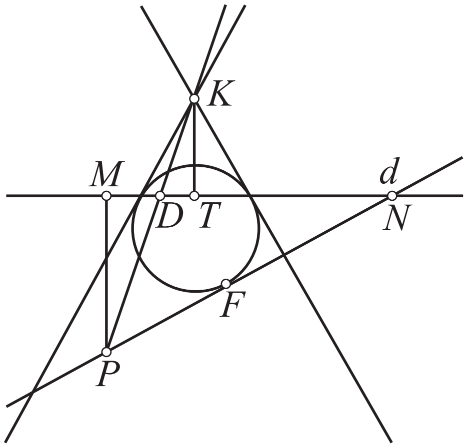
where $\Lambda = diag\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.
2. case $\exists i : \lambda_i = 0$ Let $\underline{s}_i$ be the i^{th} eigenvector such that $\left\langle \underline{s}_i, \underline{s}_j \right\rangle = \delta_{ij}$. Then $\underline{\underline{B}} = \begin{bmatrix} \underline{s}_1 & \underline{s}_2 & \dots & \underline{s}_n \end{bmatrix}$, $\underline{t} = t_1 \underline{s}_1 + t_2 \underline{s}_2 + \dots + t_n \underline{s}_n$ and $\underline{b} = b_1 \underline{s}_1 + b_2 \underline{s}_2 + \dots + b_n \underline{s}_n$. Then $\underline{\underline{A}}\,\underline{t} = \sum_{i=1}^n t_i \underline{\underline{A}}\,\underline{s}_i \Rightarrow (\underline{\underline{B}}^T (\underline{\underline{A}}\,\underline{t} + \underline{b}))_i = t_i \lambda_i + b_i$. If $\lambda_i \neq 0$, then $t_i := -\dfrac{b_i}{\lambda_i}$, otherwise t_i is arbitrary.

$n = 2,\ D = c - \underline{b}^T \underline{\underline{A}}^{-1} \underline{b}$		$D > 0$	$D = 0$	$D < 0$
$\lambda_1 > 0$	$\lambda_2 > 0$	$\emptyset$	$x = y = 0$	ellipse
$\lambda_1 > 0$	$\lambda_2 = 0$	parabola	parabola, $x^2 = 0$	parabola, $\lambda_1 x^2 = -D$
$\lambda_1 > 0$	$\lambda_2 < 0$	hyperbola	$\lambda_1 x^2 + \lambda_2 y^2 = 0$	hyperbola
$\lambda_1 = 0$	$\lambda_2 = 0$	$\emptyset$	$x, y \in \mathbb{R}$	$\emptyset$

$n=3,\ D = c - \underline{b}^T \underline{\underline{A}}^{-1} \underline{b}$			$D > 0$	$D = 0$	$D < 0$
$\lambda_1 > 0$	$\lambda_2 > 0$	$\lambda_3 > 0$	imaginary ellipsoid	origin	ellipsoid
$\lambda_1 > 0$	$\lambda_2 > 0$	$\lambda_3 = 0$	imaginary elliptic cylinder, elliptic paraboloid	imaginary intersecting planes, elliptic paraboloid	elliptic cylinder elliptic paraboloid
$\lambda_1 > 0$	$\lambda_2 = 0$	$\lambda_3 = 0$	imaginary parallel planes, parabolic cylinder	double plane, parabolic cylinder	parallel planes, parabolic cylinder
$\lambda_1 > 0$	$\lambda_2 < 0$	$\lambda_3 = 0$	hyperbolic paraboloid planes, hyperbolic cylinder	intersecting planes, hyperbolic cylinder	hyperbolic paraboloid, hyperbolic cylinder
$\lambda_1 > 0$	$\lambda_2 > 0$	$\lambda_3 < 0$	two-sheeted hyperboloid	cone	one-sheeted hyperboloid
$\lambda_1 = 0$	$\lambda_2 = 0$	$\lambda_3 = 0$	$\emptyset$	whole space	$\emptyset$

1.6 Conic sections

Definition 1.36 *A double circular cone is formed by a set of lines connecting a common point (apex) to all the points of a circle, where the orthogonal line through the center of the circle to its plane contains the apex. Intersecting a double circular cone by a plane, we obtain conic section.*



One can realize that a sphere can be inscribed by increasing its radius from K and the common points of it with the cone form a circle. Let the plane of this circle be π and our original intersecting plane be α. Let d be the intersection of π and α, the tangent point of the sphere to α be F. Let PK be an arbitrary generating line, where $P \in \alpha$ and let D be the intersection of $\overline{PK}$ and π. Finally, let M and N be the orthogonal projection of P to π and d respectively, and T be the orthogonal projection of K to π.

Since $KTD\triangle \sim PMD\triangle \Rightarrow MPD\angle = \phi$, if ϕ is half of the aperture. $|\overline{PD}| = |\overline{PF}|$, because they are tangent segments to the inscribed sphere $\Rightarrow |\overline{PM}| = |\overline{PD}| \cos\phi$. If ψ is the angle of π and α, then $|\overline{PM}| = \sin\psi|\overline{PN}| = \sin\psi|Pd| \Rightarrow \dfrac{|\overline{PF}|}{|Pd|} = \dfrac{\sin\psi}{\cos\phi} = c$.

Definition 1.37 *We say, that a conic section is ellipse, parabola, hyperbola if this ratio c is smaller than, equal to, greater than* 1 *respectively.*

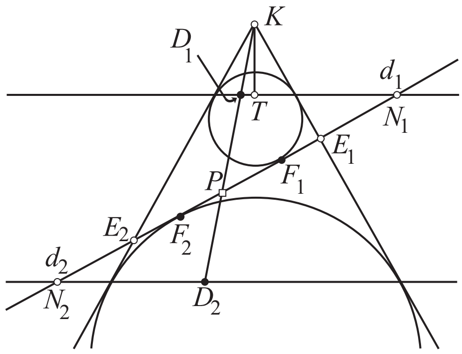


Figure 1.9: Ellipse: $c < 1$

Ellipse: Now, we assume, that α has a common point with every director line ($c < 1$). Then we have two inscribed spheres. Since $|\overline{PD_1}| = |\overline{PF_1}|$ and $|\overline{PD_2}| = |\overline{PF_2}|$, therefore $|\overline{PF_1}| + |\overline{PF_2}| = |\overline{PD_1}| + |\overline{PD_2}| = |\overline{D_1D_2}|$, but $|\overline{D_1D_2}|$ is a segment on a director line $\Rightarrow$ constant $\Rightarrow 2a := |\overline{D_1D_2}|$. For E_i, we can claim, that $|\overline{E_iF_1}| + |\overline{E_iF_2}| = 2a$ (tangent segments). $2c := |\overline{F_1F_2}| \Rightarrow 2a = |\overline{E_1F_1}| + |\overline{E_1F_2}| = 2|\overline{E_1F_1}| + |\overline{F_1F_2}| \Rightarrow |\overline{E_1F_1}| = a - c$. Similarly, $|\overline{E_2F_2}| = a - c \Rightarrow |\overline{E_1E_2}| = 2a$.

Definition 1.38 *Let F_1 and F_2 be two points on a plane such that $d(F_1, F_2) = 2c$ and a be a real number such that $a > c$. Then the locus of points for which the sum of the distances to F_1 and F_2 is $2a$ is called ellipse.*

Parabola: Let α be parallel to exactly one director line ($c = 1$). Then there is no more Dandelin sphere and $|\overline{PF}| = |Pd|$.

Definition 1.39 *Let F be a point and d be a line, not lying on the point. Then the locus of points for which the distance to the point is equal to the distance to the line is equal is called parabola.*

Hyperbola: Let α be parallel to exactly one director line ($c > 1$). Then we have two inscribed spheres, one in each part of the double cone. Similarly to the case of the ellipse, $|\overline{D_1D_2}| = |\overline{E_1E_2}| =: 2a$ and $|\overline{F_1F_2}| =: 2c$, but now $c > a$. It is easy to prove, that $\left||\overline{PF_1}| - |\overline{PF_2}|\right| = 2a$

Definition 1.40 *Let F_1 and F_2 be two points on a plane such that $d(F_1, F_2) = 2c$ and a be a real number such that $a < c$. Then the locus of points for which the difference of the distances to F_1 and F_2 is $2a$ is called hyperbola.*

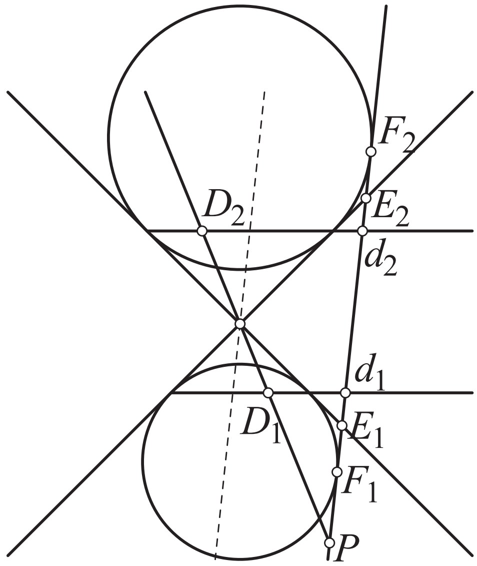


Figure 1.10: Hyperbola: $c > 1$

1.6.1 Tangents from external point to conic sections

Ellipse: Let F_1, F_2 be the focis, a be the semi-mayor axis, $b := \sqrt{a^2 - c^2}$ be the semi-minor axis and K be an external point.

Lemma 1.41 *If we draw a circle around F_2 with radius $2a$ then the E_1 reflection of F_1 in any tangent line of the ellipse lies on this circle.*

Proof. Let T be the point of tangency, then $|\overline{F_1T}| = |\overline{E_1T}|$ and the tangent t is the perpendicular bisector of $\overline{E_1F_1}$. Suppose, that T is not on the line of E_1F_2. Then because of the triangle inequality $2a = |\overline{TF_1}|+|\overline{TF_2}| = |\overline{TE_1}|+|\overline{TF_2}| \geq |\overline{E_1F_2}|$. But then $E_1F_2 \cap t \neq \emptyset \Rightarrow Q := E_1F_2 \cap t \Rightarrow 2a \geq |\overline{QE_1}| + |\overline{QF_2}| = |\overline{QF_1}| + |\overline{QF_2}| \Rightarrow Q$ is inside the ellipse and $t = TQ$ is not a tangent, therefore $T = Q$ and $2a = |\overline{TE_1}| + |\overline{TF_2}| = |\overline{E_1F_2}| \Rightarrow E_1$ lies on the circle. ■

Construction. Let t_1 and t_2 be the tangent of the ellipse through K, and the reflection of F_1 in them be E_1 and E_2 respectively. Then $|\overline{KE_1}| = |\overline{KF_1}| = |\overline{KE_2}|$, since t_i is the perpendicular bisector of $\overline{F_1E_i}$. Then E_1 and E_2 lies on the circle around K with radius $|\overline{KF_1}|$ and on the circle around F_2 with radius $2a$. $\Rightarrow t_i$ can be constructed. ■

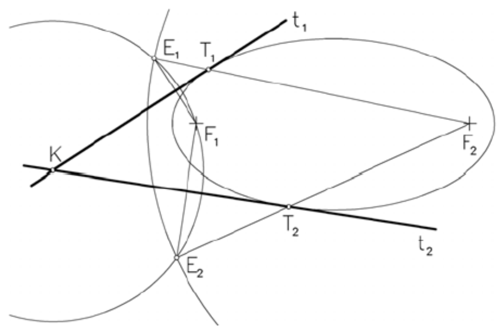


Figure 1.11: Tangents to an ellipse

Parabola: Let F be the foci, d be the directrix and K be an external point.

Lemma 1.42 *The E reflection of F in a tangent line lies on the directrix.*

Proof. Let T be the point of tangency, then the directrix is tangential to the circle around T with radius $|\overline{TF}| \Rightarrow d \perp TQ$, where Q is the common point of the circle and the directrix. Then T lies on the perpendicular bisector b of $\overline{FQ}$. Assuming, that b has another common point S with the parabola, we get, that $|\overline{QS}| = |\overline{FS}| = dist(S, d)$, but Q lies on d. Let U be the footpoint of the perpendicular line to d through S, then $UQS\triangle$ is an isosceles triangle with two right angle $\Rightarrow b$ has only one common point T with the parabola $\Rightarrow b$ is the tangent to $T \Rightarrow b = t$ and $E = Q$. ■

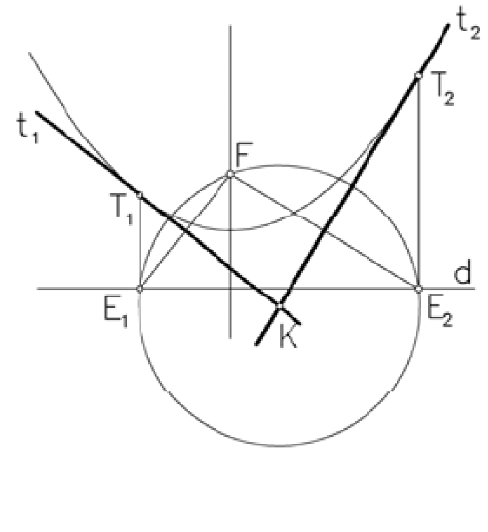


Figure 1.12: Tangents to a parabola

Construction. Let t_1 and t_2 be the tangent of the parabola through K, and the reflection of F in them be E_1 and E_2 respectively. Then $|\overline{KE_1}| = |\overline{KF}| = |\overline{KE_2}|$, since t_i is the perpendicular bisector of $\overline{FE_i}$ and E_i lies on the circle around K with radius $|\overline{KF_1}|$ and on the directrix. $\Rightarrow t_i$ can be constructed. ■

Hyperbola: The construction process is similar to the case of the ellipse.

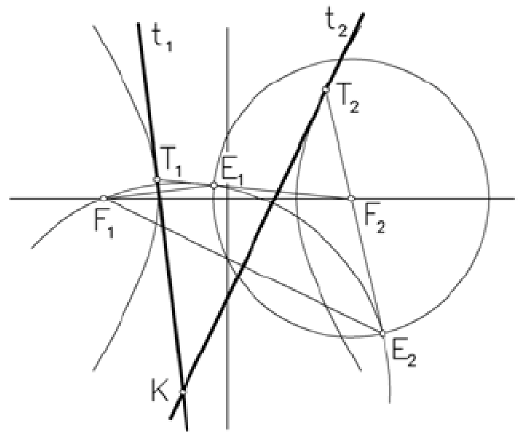


Figure 1.13: Tangents to a hyperbola

1.7 Polyhedrons in n-dimensional Euclidean space

Definition 1.43 *$X = \underline{x}_0 + V_k$ is a k-dimensional affine subspace, if V_k is a real k-dimensional real subspace and $\underline{x}_0 \in \mathbf{E}^n$. If $k = n-1$, X is called hyperplane and can be represented as $\langle \underline{x}, \underline{n} \rangle = c$, where $\underline{n} \in V_{n-1}^{\perp}$. Every hyperplane divides the space into three parts: $H := \{\underline{x} | \langle \underline{x}, \underline{n} \rangle = c\}$, $H^- := \{\underline{x} | \langle \underline{x}, \underline{n} \rangle > c\}$, $H^+ := \{\underline{x} | \langle \underline{x}, \underline{n} \rangle < c\}$.*

Definition 1.44 *Let H be a hyperplane, then $H \cup H^+$ is a closed hyperplane.*

Definition 1.45 *C is an n-dimensional convex convex polyhedron, if it is bounded and can be obtained by the intersection of finitely many closed hyperplanes such that it contains an n-dimensional sphere.*

Definition 1.46 $H_i \cup H_i^+$ *is a crucial hyperplane of* C*, if* $C \cap H_i \neq \emptyset$ *and* $\exists Q \in C \cap H_i \forall j \neq i : Q \notin H_j$.

Definition 1.47 *Let* C *be an* n*-dimensional convex polyhedron, then the* $(n-1)$*-dimensional facets of* C *are the intersections of* C *with its crucial hyperplanes.*

Lemma 1.48 *The* $(n-1)$*-dimensional facets of an* n*-dimensional convex polyhedron are* $(n-1)$*-dimensional convex polyhedrons.*

Proof. Let Q be a point in C such that H_i is the closest hyperplane to Q and $d := \min_{i \neq j}\{dist(Q, H_j)\}$. Let G be an n-dimensional sphere around Q with radius d, then $G \cap H_i$ is an $(n-1)$-dimensional sphere inside $C \cap H_i$ and $C \cap H_i = \bigcap_{i \neq j}(H_i \cap (H_j \cup H_j^+)) \Rightarrow C \cap H_i$ is an $(n-1)$-dimensional polyhedron . ■

Remark 1.49 *We can construct a chain of polyhedrons. The facets of any* k*-dimensional polyhedron are* $(k-1)$*-dimensional polyhedrons.*

Definition 1.50 *The 1-dimensional polyhedrons are edges and the 0-dimensional polyhedrons are vertices.*

Theorem 1.51 *Let* C *be an* n*-dimensional convex polyhedron with* $\{V_1, V_2, \ldots V_m\}$ *vertices. Then* C *is the convex hull of* V_i*:* $C = \{\sum \alpha_i \underline{v}_i | \alpha_i \geq 0 \wedge \sum \alpha_i = 1\}$.

Proof. Induction For $k = 0$, it is true and for $k = 1$, we get edges $x = \alpha \underline{v}_i + (1-\alpha)\underline{v}_j$. Now let P be a point on a $(k+1)$-dimensional hyperface. Let v_i be a vertex of this hyperface and $\overrightarrow{V_iP}$ ray intersects it in Q. If Q is a vertex, then P lies on an edge and we are done, otherwise Q lies on a k-dimensional hyperplane $\Rightarrow \underline{q} = \sum_{i=1}^{l} \alpha_i \underline{v}_i$, where $\alpha_i \geq 0$ and $\sum \alpha_i = 1$. Since $P \in \overline{QV_i} : \underline{p} = \alpha \underline{v}_i + \beta \underline{q} = \alpha \underline{v}_i + \beta \sum \alpha_j \underline{v}_j = \sum_{i \neq j} (\alpha_j \beta) \underline{v}_j + (\alpha + \beta \alpha_i) \underline{v}_i$, where $\alpha + \beta = 1 \Rightarrow \alpha + \beta(\sum \alpha_j) = 1$. ■

1.7.1 3-dimensional polyhedrons

Definition 1.52 S *is a star-shaped polyhedron, if any edge belongs to exactly two facets, the facets and the whole surface is simply connected, edge connected, face connected and* $\exists P \in S \forall X \in S : \overline{PX} \subset S$.

Theorem 1.53 (Euler) *Let* v*,* e *and* f *be the number of vertices, edges and facets in a star-shaped polyhedron. Then* $v + f = e + 2$.

Proof. If we project the polyhedron onto a sphere around an appropriate point, then the structure will not change. Now, we chose an inside point of a spherical facet and we apply a stereographic projection to get a planar graph. If there is a circle in it, then erasing an edge of it will result in a planar graph with -1 domain and -1 edge. When there is not any circle in the graph, then we get a tree with v vertices and only 1 domain. This tree has $v-1$ edges $\Rightarrow (v-1)+2 = v+1$. Then $e+2 = v+f$ was true at the beginning. ∎

Remark 1.54 *If we are in n-dimension, then $\sum_{i=0}^{n}(-1)^i f_i = 1+(-1)^{n-1}$, where f_i is the number of the i-dimensional facets.*

Definition 1.55 *The union of regularly connected simple polygons is called polyhedral surface if any edge belongs to at most two polygons. The boundary of a polyhedral surface is the union of all the edges which belong to exactly one polygon. A polyhedral surface is closed, if the boundary of it is an empty set.*

Definition 1.56 *A cycle on a polyhedral surface is a closed series of edges, such that every vertex appears exactly twice.*

Lemma 1.57 *A closed polyhedral surface is simply connected, if for every circle, there exists an F set of facets, such that $\forall p \in F, p \notin F \forall p = e_0, e_1, \ldots, e_n = p'$ edge series $\exists i : f_i$ belongs to the cycle.*

Remark 1.58 *The Euler theorem is true for any face connected, simply connected closed polyhedral surface.*

Definition 1.59 *Let C be a convex polyhedron and V be a vertex of it. Then the vertex figure of V is formed by the points, lying on the surface of the unit ball around V and on the edge rays, emanating from V.*

Definition 1.60 *Any C polyhedron is regular, if and only if all of its facets are congruent regular polygons and all of its vertex figures are congruent regular polygons.*

Theorem 1.61 *There exists exactly five regular polyhedron in 3D.*

Proof. Let the facets be regular n-gons and the vertex figures be regular m-gons. Then $2e = f \cdot n = v \cdot m \Rightarrow f = \frac{2e}{n}$ and $v = \frac{2e}{m}$, but according to the Euler theorem: $f + v = \frac{2e}{n} + \frac{2e}{m} = 2 + e \Rightarrow \frac{1}{n} + \frac{1}{m} = \frac{1}{e} + \frac{1}{2} > \frac{1}{2}$ and $n, m \geq 3 \Rightarrow (m,n) \in \{(3,3),(3,4),(4,3),(3,5),(5,3)\}$. Now, we consider the dual of a regular polyhedron as follows. The vertices of the dual polyhedron is the center of the facets, and we connect to vertex if and only if the facets are neighboring.

$(3,3) \Rightarrow$ tetrahedron

$(4,3) \Rightarrow$ cube $\Rightarrow$ its dual is the octahedron

$(5,3) \Rightarrow$ dodecahedron: vertices: $(\pm 1, \pm 1, \pm 1)$, $\left(0, \pm\frac{2}{1+\sqrt{5}}, \pm\frac{1+\sqrt{5}}{2}\right)$, $\left(\pm\frac{2}{1+\sqrt{5}}, \pm\frac{1+\sqrt{5}}{2}, 0\right)$ and $\left(\pm\frac{1+\sqrt{5}}{2}, 0, \pm\frac{2}{1+\sqrt{5}}\right)$ $\Rightarrow$ its dual is the icosahedron. ■

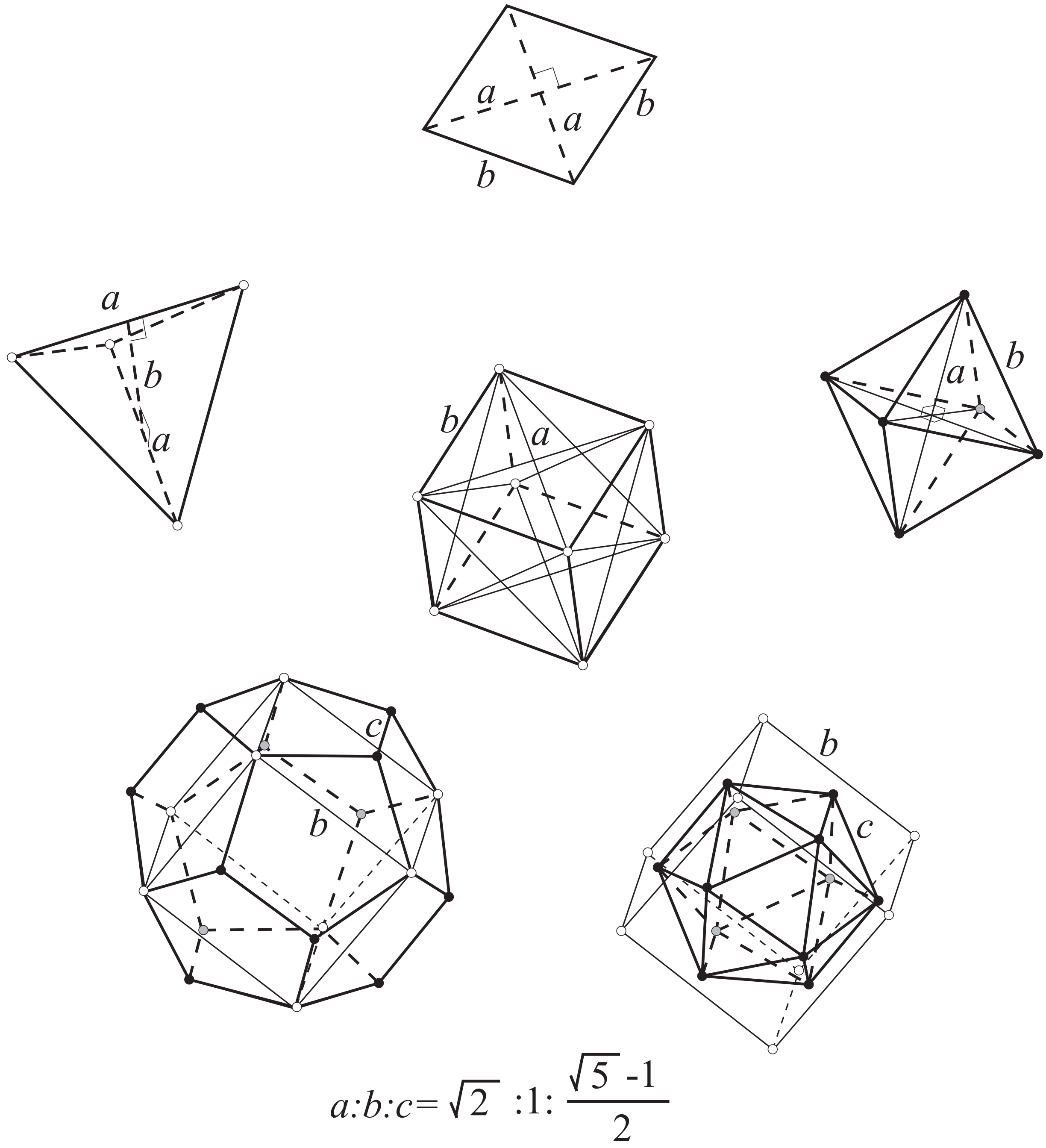


Figure 1.14: Regular polyhedrons

1.8 Congruence of polyhedra

Definition 1.62 *The face lattice of a polyhedron is a partially ordered set, which consists of the vertices, edges and facets, such that order is provided by the set theory containment. Two polyhedra are combinatorically equivalent if their face lattice are isomorphic, i.e, there exists a bijection between vertices, edges and facets such that it preserves order.*

Theorem 1.63 (Cauchy's rigidnes theorem) *If, two convex polyhedra are combinatorically equivalent, and each pair of facets are congruent to each other, then the polyhedrons are congruent.*

Lemma 1.64 (Spherical Arm-lemma) *Let A_i and A'_i be the vertices of two spherical polygon such that $\forall i = 1, 2, \ldots, n-1 : \overline{A_iA_{i+1}} \cong \overline{A'_iA'_{i+1}}$ and $\forall i = 2, \ldots, n-1$: $A_{i-1}A_iA_{i+1}\angle \leq A'_{i-1}A'_iA'_{i+1}\angle$. Then $\overline{A_1A_n} \leq \overline{A'_1A'_n}$ holds.*

Proof. Induction: For $n = 3$, it is the spherical triangle inequality $\Rightarrow$ true.
Case 1: $\exists i : A_{i-1}A_iA_{i+1}\angle = A'_{i-1}A'_iA'_{i+1}\angle$
Case 2: $\forall i : A_{i-1}A_iA_{i+1}\angle < A'_{i-1}A'_iA'_{i+1}\angle$, but $A_{n-2}A_{n-1}A_n\angle$ can be increased to $A'_{n-2}A'_{n-2}A'_n\angle$, such that $A_1, A_2, \ldots, \widetilde{A}_n$ remains convex.
Case 3: $\forall i : A_{i-1}A_iA_{i+1}\angle < A'_{i-1}A'_iA'_{i+1}\angle$, but $A_{n-2}A_{n-1}A_n\angle$ cannot be increased to $A'_{n-2}A'_{n-2}A'_n\angle$, because $A_2A_1\overline{A}_n$ are collinear points, $A_{n-2}A_{n-1}\widetilde{A}_n\angle = A'_{n-2}A'_{n-2}A'_n\angle$, but $A_1, A_2, \ldots, \tilde{A}_n$ is non-convex.
Case 1-2: One vertex can be deleted, either A_i or A_{n-1}.
Case 3: We compare $A_2, \ldots, A_{n-1}, \overline{A}_n$ $(n-1)$-gon to $A'_2, \ldots, A'_n$ $(n-1)$-gon. Then $\overline{A_2\overline{A}_n} \leq \overline{A'_2A'_n}$, by induction, but $\overline{A_2A_1} + \overline{A_1\overline{A}_n} = \overline{A_2\overline{A}_n} \leq \overline{A'_2A'_n} \leq \overline{A'_2A'_1} + \overline{A'_1A'_n}$ and $\overline{A'_2A'_1} = \overline{A_2A_1} \Rightarrow \overline{A_1A_n} \leq \overline{A'_1A'_n}$.
■

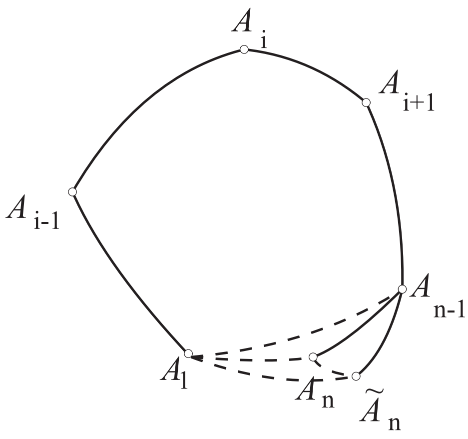


Figure 1.15: Case 2

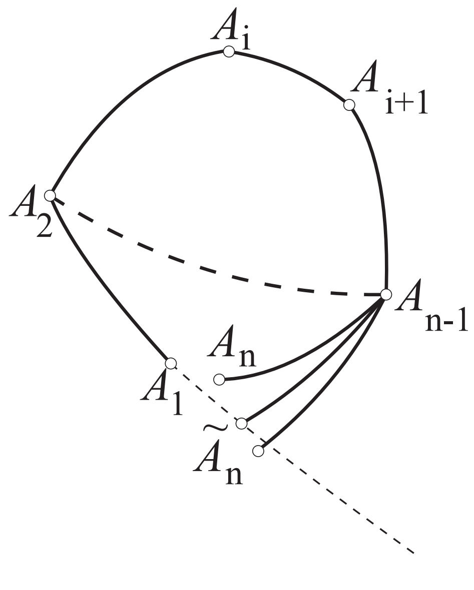


Figure 1.16: Case 3

Assume, that $A_1, \ldots, A_n$ and $A'_1, \ldots, A'_n$ are spherical polygons, such that $\overline{A_iA_{i+1}} = \overline{A'_iA'_{i+1}}$. If $A_{i-1}A_iA_{i+1}\angle > A'_{i-1}A'_iA'_{i+1}\angle$, then we put $+$ to A_i, if $A_{i-1}A_iA_{i+1}\angle < A'_{i-1}A'_iA'_{i+1}\angle$, then we put $-$ to A_i, otherwise we put 0 to A_i.
There is a sign change, if $A_i = \pm$ and $A_{i+j} = \mp$ if $\forall k = 1, \ldots, j-1 : A_{i+k} = 0$.

Lemma 1.65 (Sign lemma) *If, there is a sign on a polygon, then there are at least 4 sign changes.*

Proof. The number of sign changes is even. If, there is a sign, then there ought to be another different sign, otherwise we have a contradiction by our previous lemma. Two sign changes can only be happen if + and − signs are in two blocks. In the two sub-polygons, we apply the spherical arm lemma. We get that the diagonal is smaller and greater in $A'_1, \ldots, A'_n$ than in $A_1, \ldots, A_n$ at the same time. ∎

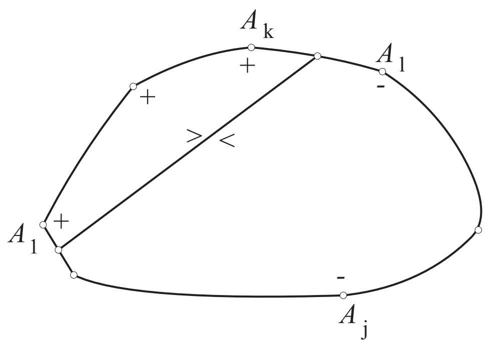


Figure 1.17: Sign lemma

Proof. (Cauchy-theorem)

Case 1: If all the dihedral angles are the same, then we build up the polyhedrons face-by-face.

Case 2: If all the dihedral angles change, then we write + or − if the dihedral angle is greater or smaller in P' than in P. We can apply the sign lemma on the surface of a small sphere around every vertex. Sign change: Two edges, sharing a vertex on a face have different signs $\Rightarrow 4v \leq \delta$, where δ is the number of sign changes. But on the faces $\delta \leq \sum_{k\geq 3} 2 \left[\frac{k}{2}\right] \alpha_k \leq \sum 2(k-2)\alpha_k = 2\sum k \cdot \alpha_k - 4\sum \alpha_k = 4e - 4f \Rightarrow 4v \leq \delta \leq 4v - 4f$, but $4v + 4f = 4e + 8$, so this is a contradiction.

Case 3: If some of the dihedral angels change, then a vertex is 'real', if it has an edge with a sign. We delete the ghost edges and the new facets are topologically connected surfaces. Then $v + f \leq e$. We add the ghost edges back one-by one, if one of its endpoint is in the graph. With every possibility, the $v + f \leq e$ inequality remains true $\Rightarrow$ contradiction. ∎