Area and volume

2.1 Area on the Euclidean plane

Definition 2.1 *Area is an isometry invariant, additive, non-negative set function for simple polygons and 1 is assigned to the unit square.*

Lemma 2.2 *The area of the rectangle is the product of the length of its neighboring sides.*

Lemma 2.3 *The area of the triangle is the half of the product of the length of one of its side and the corresponding altitude.*

Proof. We cut the triangle int two right angled triangle by one of the altitudes. Another copies of these triangles form two rectangles, where the sum of the appropriate sides is the side of the triangle and the other is the corresponding altitude. ■

Lemma 2.4 *Any simple polygon can be divided into triangles by non-intersecting diagonals.*

Proof. INDUCTION: For $n = 3$ it is trivial.
Consider the leftmost point v of the polygon and its neighbors u and v. If $\overline{uv}$ is a diagonal, then we are done. Otherwise, there is at least on vertex of the polygon inside $uvw\triangle$. Consider the furthermost of them to $\overline{uv} \Rightarrow z$. Then $\overline{vz}$ is a diagonal. ■

Remark 2.5 *We get exactly $n - 2$ triangles.*

Definition 2.6 *The area of any simple poygon is the sum of the areas of the triangles, of which the polygon consists.*

Definition 2.7 *A set H has area, if the infimum of the area of the circumscribed polygons is equal to the supremum of the area of the inscribed polygons.*

Lemma 2.8 *Area is a monotonic function, if $A \subseteq B$ and both have area, then $A(A) \leq A(B)$.*

Theorem 2.9 *Every convex bounded set has area.*

2.2 Area on the hyperbolic plane

Definition 2.10 *A triangle is called asymptotic, if one of its vertex is a boundary point. A triangle is called doubly/triply asymptotic, if two/three of its vertices are boundary points.*

Theorem 2.11 *All the triply asymptotic triangles are congruent to each other.*

Definition 2.12 *Hyperbolic area is an isometry invariant, additive, non-negative set function for simple polygons and π is assigned to the triply asymptotic triangle.*

Theorem 2.13 *Any asymptotic triangle can be cut off into a pentagon.*

Proof. (Poincaré disk model): Let $ABC\triangle$ be such that C is the ideal point and A is the center of the model. Let D be an ideal point such that (ABD) and M be the footpoint of the perpendicular line to CD through A. Let A_1 be the reflection of B in the line AM and the intersection of BC and A_1D be M_1. Let the footpoints of the perpendicular line to DC through B and A_1 be Q and P respectively. If M_2 is the intersection of A_1P and BC and the reflection of BC in A_1P is DA_2, then $M_2A_1A_2\triangle$ is congruent to $A_1M_1M_2\triangle$ and $M_1M_2B\triangle$. Continuing this procedure, we get the $ABQPA_1$ pentagon. ■

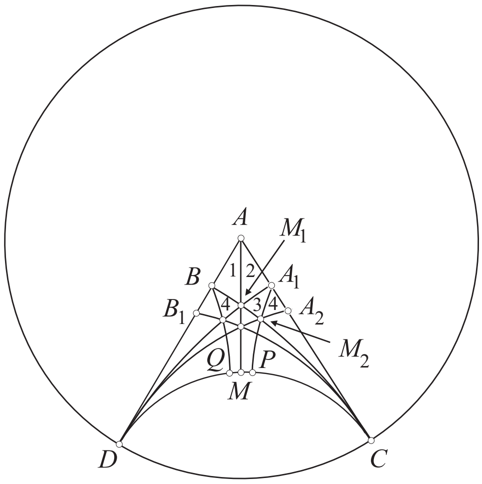

Figure 2.1: Lendin construction

Theorem 2.14 *If, the angle of a doubly asymptotic triangle at the proper vertex is α, then the area of this triangle is $(\pi - \alpha)$.*

Proof. Let $f(\phi)$ be the area, if $\phi = \pi - \alpha$. is the supplementary angle. The union of the two corresponding doubly asymptotic triangle is a triply asymptotic triangle: $\pi = f(\phi) + f(\pi - \phi)$ A triply asymptotic triangle can be cut off into three doubly asymptotic triangle (see Figure 2.2): $\pi = f(\phi) + f(\psi) + f(\pi - \phi - \psi)$. Using the previous result for $\phi + \psi$ we obtain that $f(\phi) + f(\psi) = f(\phi + \psi)$. Our only solution is $f(x) = \lambda x$, since f is monotonously increasing and if $f(1) = \lambda \Rightarrow f(n) = n \cdot \lambda$. Now, if $\frac{k}{n} \leq x \leq \frac{k+1}{n} \Rightarrow k \leq nx \leq k+1 \Rightarrow f(k) \leq f(nx) \leq f(k+1) \Rightarrow \lambda k \leq nf(x) \leq \lambda(k+1) \Rightarrow \frac{k}{n} \leq \frac{f(x)}{\lambda} \leq \frac{k+1}{n} \Rightarrow \forall n : \left| x - \frac{f(x)}{\lambda} \right| \leq \frac{1}{n} \Rightarrow f(x) = \lambda x$. Finally $\pi = f(0) + f(\pi) = f(\pi) \Rightarrow \lambda = 1$. ■

Theorem 2.15 *The area of any hyperbolic triangle is its defect.*

Proof. Let D, E and F be boundary points, such that (ABD), (BCE) and (CAF). Then the area of $DEF\triangle = \pi$ but

$$\pi = area(FDE\triangle) = area(ABC)+area(ADF)+area(BDE)+area(CEF) = area(ABC)+\alpha+\beta+\gamma$$

Therefore $area(ABC) = \pi - (\alpha + \beta + \gamma)$. ■

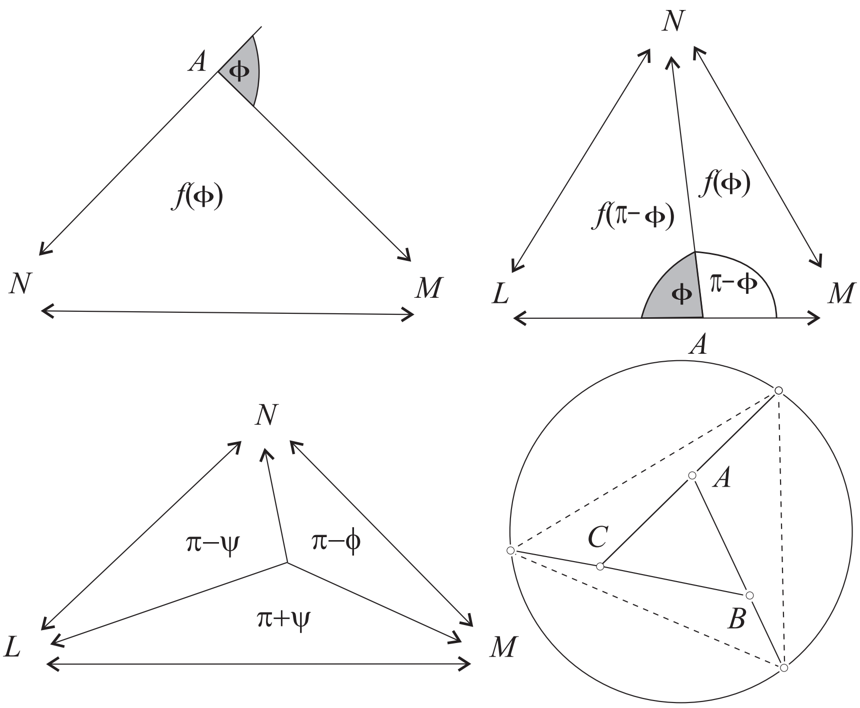

Figure 2.2: Hyperbolic area

2.3 Volume in the Euclidean space

Area: We only used the principum that two polygons have the same area if and only if the first can be cut into finitely many polygonal pieces that can be reassembled to yield the second.

Question: Can we extend this to 3D? (NO!)

Definition 2.16 *Two polyhedrons are scissors-congruent if the first can be cut into finitely many polyhedral pieces that can be reassembled to yield the second.*

Dehn-invariant: We assign a value to every polyhedron such that two scissors-congruent polyhedrons have the same value and $D(\mathcal{P}) = \sum_{i=1}^{n} D(\mathcal{P}_i)$ if $\mathcal{P}$ has been cut into $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$. Let f be an additive function such that $f(0) = f(\pi) = 0$, $l(e)$ be the length of the edge e and Θ_e be the dihedral angle between the two facets meeting at e. Then $D(\mathcal{P}) := \sum_{e \in edges} f(\Theta_e) l(e)$

- If $e \in \mathcal{P}_k$ and e is inside $\mathcal{P}$, then the sum of the dihedral angles around e is $2\pi \Rightarrow f(2\pi)l(e) = 2f(\pi)l(e) = 0$.

- If $e \in \mathcal{P}_k$ and e belongs to a facet of P, then the dihedral angles around e is $\pi \Rightarrow f(\pi)l(e) = 0$.

- If $e \in \mathcal{P}_k$ and e belongs to the e' edge of $\mathcal{P}$, then $f(\Theta_{e'})l(e)$

Theorem 2.17 *The regular tetrahedron $\mathcal{T}$ and the cube $\mathcal{C}$ are not scissors-congruent.*

Proof. Let $l(e) := 1$ for every e edge in $\mathcal{T}$. Then $D(\mathcal{T}) = 6f(\Theta)$, where $\Theta = \arccos\left(\frac{1}{3}\right)$ is the dihedral angle of the regular tetrahedron. It is known, that neither Θ nor π is rational $\exists f$ additive function such that $f(\Theta) = 1$ and $f\left(\frac{\pi}{2}\right) = 0$. Since Θ and π are independent vectors in $\mathbb{R}^1$ over $\mathbb{Q} \Rightarrow \exists$ base with Θ and π. Let $f(\Theta)$ be 1 and $f(b)$ be 0 for all $b \in$base (f is not linear over $\mathbb{R}$). Then f satisfies all conditions and $D(\mathcal{T}) = 6$ but $D(\mathcal{C}) = 0$. ■

Cavalieri principle: The set function V satisfies the Cavalieri principle if the following property is true for any H_1 and H_2 sets in the domain of V: If every element of a parallel plane pencil intersects both H_1 and H_2 in cross-sections of equal are, then $V(H_1) = V(H_2)$.

Definition 2.18 *Volume is an isometry invariant, non-negative, additive set function for simple polyhedrons, which satisfies the Cavalieri principle and 1 is assigned to the unit cube.*

Theorem 2.19 $V(\mathcal{T}) = \frac{1}{3}area(ABC_{\triangle}) \cdot m$*, where* $m = dist(D, ABC)$.

Proof.

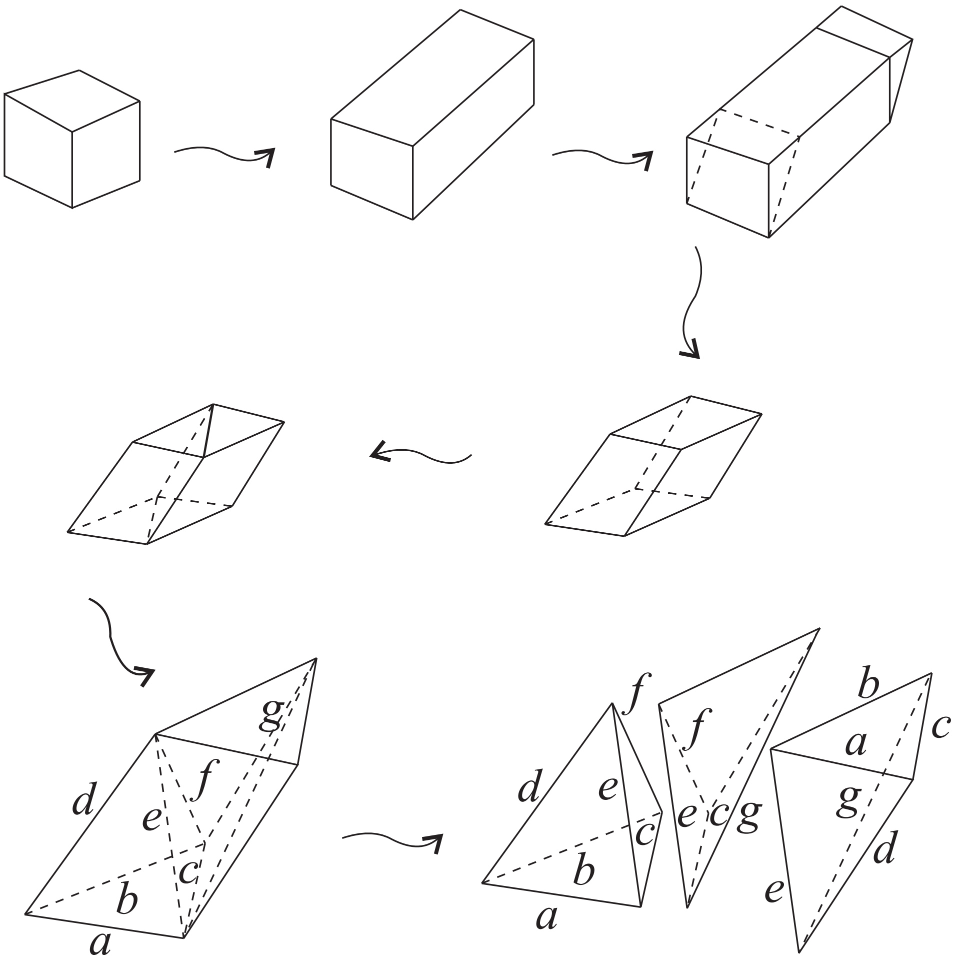

Figure 2.3: Volume of the tetrahedron

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