LECTURE NOTES IN

GEOMETRY

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1 Absolute geometry

1.1 Axiom system

We will follow the axiom system of Hilbert, given in 1899. The base of it are primitive terms and primitive relations. We will divide the axioms into five parts:

1. Incidence
2. Order
3. Congruence
4. Continuity
5. Parallels

1.1.1 Incidence

Primitive terms: point, line, plane
Primitive relation: "lies on"(containment)

Axiom I1 *For every two points there exists exactly one line that contains them both.*

Axiom I2 *For every three points there exists exactly one plane that contains all of them.*

Axiom I3 *If two points of a line lie in a plane, then every point of the line lies in the plane.*

Axiom I4 *If two planes have a point in common, then they have another point in common.*

Axiom I5 *There exists four points not lying in a plane.*

Remark 1.1 *The incidence structure does not imply that we have infinitely many points. We may consider 4 points* $\{A, B, C, D\}$*. Then, the lines can be the two-element subsets and the planes the three-element subsets.*

1.1.2 Order

Primitive relation: "betweeness"
Notion: If A, B, C are points of a line then $(ABC) := B$ means the "B is between A and C".

Axiom Os1 *If* (ABC) *then* (CBA).

Axiom Os2 *If* A *and* B *are two points of a line, there exists at least one point* C *on the line* AB *such that* (ABC).

Axiom Os3 *Of any three points situated on a line, there is no more than one which lies between the other two.*

Axiom Os4 (Pasch) *Let* A, B, C *be three points not lying on the same line and let* e *be a line lying in the plane* ABC *and not passing through any of the points* A, B, C. *Then, if the line* e *passes trough a point* D *such that* (ADB), *it will also pass trough a point* E *such that either* (BEC) *or* (AEC).

Remark 1.2 *The first three order axioms do not implies that the line has infinitely many points. Let a model of the line be the edge of a regular pentagon. We define the betweeness such that* (XYZ) *if and only if the triangle of* X, Y, *and* Z *is an isosceles triangle with* Y *as edge vertex.*

Lemma 1.3 *In the Pasch axiom, either* (BEC) *or* (AEC) *but not both at once.*

Proof. INDIRECT: $(AEC) \wedge (BFC) \Rightarrow (DEF)$

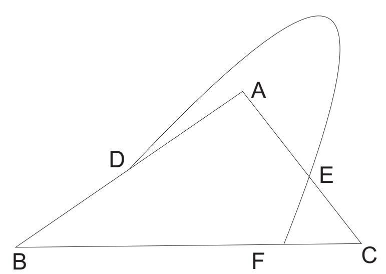


Figure 1.1: Proof: Lemma 1.3

We consider the Pasch axiom for the non-collinear points D, B, F and the line of AC (see Figure 1.1). Therefore either (BAD) or (BCF), but we supposed in the axiom that (ADB) and in the indirect assumption that (BFC). ■

Definition 1.4 *Let* A *and* B *be two points. The segment* $\overline{AB}$ *consists of those points of* C, *for which* (ACB).

Lemma 1.5 *Any segment has a point.*

Proof.

1. C is not on the line of AB
2. Let D be such that (ACD) (see Axiom Os2)
3. Let E be such that (DBE) (see Axiom Os2)
4. We use the Pasch axiom for the points A, D, B and the line $CE \Rightarrow \exists F : (AFB)$

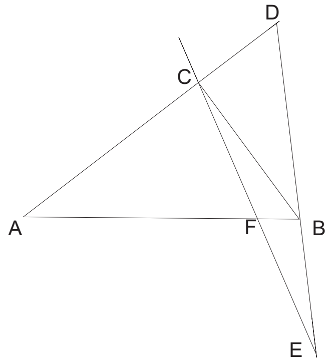


Figure 1.2: Proof: Lemma 1.5

■

Corollary 1.6 *We can define polygonal chain, polygon, half-plane, ray, angle and convexity.*

Theorem 1.7 (Jordan) *Any simple, closed polygon splits the plane into two parts. One of the parts does not contain rays t this is called the inside of the polygon.*

1.1.3 Congruence

Primitive relation: "congruence of segments and angles"

Axiom C1 *The congruence of segments and angles are equivalence relation.*

Axiom C2 *Every segment can be laid off upon a given side of a given point of a given line.*

Axiom C3 *Let $\overline{AB}$ and $\overline{BC}$ be two segments of a line which have no points in common aside from B and let $\overline{A'B'}$ and $\overline{B'C'}$ be two segments of the same or of another line, likwise no point other than B' in common. Then, if $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, we have $\overline{AC} \cong \overline{A'C'}$.*

Axiom C4 *An angle congruent to a given angle can be added to a given ray of a given line of a given plane in both half-planes determined by the given line.*

Axiom C5 *If, in two triangles ABC and $A'B'C'$ the congruences $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$ and $BAC\angle \cong B'A'C'\angle$ hold, then the congruence $\overline{BC} \cong \overline{B'C'}$ also holds.*

Remark 1.8 *With congruence we can*

- *compare segments and angles.*
- *introduce "length" like an additive non-negative, real function.*
- *introduce right angle like an angle divides the plane into four congruent parts.*
- *use the Descartes coordinate system with analytic geometry.*

1.1.4 Continuity

Axiom 1.9 (Archimedes) *If $\overline{AB}$ and $\overline{CD}$ are any segments then there exists an integer number n such that n segments $\overline{C_nD_n}$ constructed contiguously from A, along the ray from A through B, will pass beyond the point B.*

Axiom 1.10 (Cantor) *If $\overline{P_iQ_i}$ $i = 1 \ldots \infty$ are closed segments of a line such that $\overline{P_iQ_i} \supseteq \overline{P_jQ_j}$ if $j > i$, then $\bigcap_{i=1}^{\infty} \overline{P_iQ_i} \neq \emptyset$.*

Axiom 1.11 (Dedekind) *For every partition of all the points on a line into two nonempty sets such that no point of either lies between two points of the other, there is a point of one set which lies between every other point of that set and every point of the other set.*

Theorem 1.12 *The Axioms 1.9 and 1.10 are equivalent with the Axiom 1.11.*

Proof. ONLY $(D) \Rightarrow (C)$
Let $A := \{S \in e | \exists i(SP_iQ_i)\}$ and $B := \{T \in e | \exists i(P_iQ_iT)\}$. Using the axiom, we get a point C such that $\forall S \in A$ and $\forall T \in B : (SCT) \Rightarrow (\forall i : C \in \overline{P_iQ_i})$. ■

Remark 1.13 *Rational number line satisfies (A) but not (C).*

Question: Exists such model of the line that satisfies incidence, order and congruence but not (A)?
Answer: YES (Veronese)
We introduce ordering as follows:
$(x_1, y_1) > (x_2, y_2) \iff y_1 > y_2 \vee ((y_1 = y_2) \wedge x_1 > x_2)$
A point (x', y') is between (x_1, y_1) and (x_2, y_2) if and only if either $(x_1, y_1) < (x', y') < (x_2, y_2)$ or $(x_1, y_1) > (x', y') > (x_2, y_2)$.

Remark 1.14 *This model do not satisfies (C), e.g. $P_i := (n, 0)$, $Q_i := (-n, 1)$.*

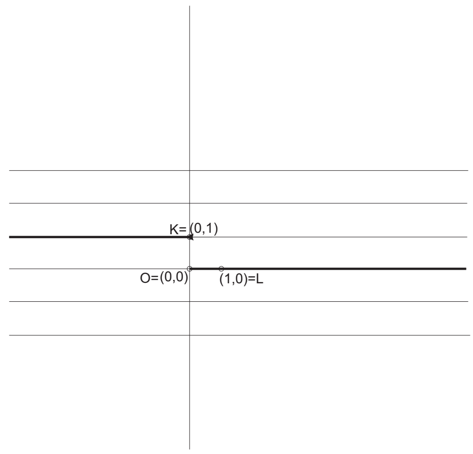


Figure 1.3: Non-Archimedean line model

1.1.5 Parallels

Axiom 1.15 (EUC) *Let a be any line and A a point not on it. Then, there is exactly one line in the plane, determined by a and A, that passes through A and does not intersect a.*

Axiom 1.16 (HYP) *Let a be any line and A a point not on it. Then, there are at least two lines in the plane, determined by a and A, that pass through A and do not intersect a.*

1.2 Absolute theorems

Lemma 1.17 *If we connect the points A and B which lie on different rays of an angle, then inside rays emanating from its vertex O intersect the segment $\overline{AB}$.*

Proof. Let C be such that (AOC) /Axiom Os2/. We apply the Pasch axiom for the points A, B, C and the ray. ■

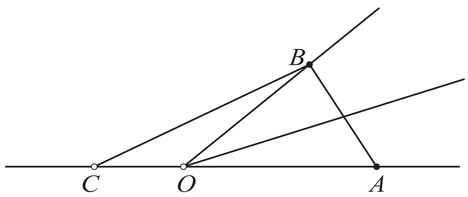


Figure 1.4: Proof: Lemma 1.17

Lemma 1.18 (External angle) *Any external angle of a triangle is larger than its any internal angle not beside it.*

Proof. Indirect $CAB\angle > CBD\angle$ where D is such that (ABD).
We lay off the $CBD\angle$ upon the $CAB\angle \Rightarrow M$ by using 1.17 lemma. Let M' be such that (CBM') and $\overline{BM} \cong \overline{M'B}$. Then $ABM'\angle = MBD\angle = MAB\angle \Rightarrow ABM'\triangle \cong BAM\triangle$. Therefore $M'AB\angle \cong ABM\angle \Rightarrow M$, A, M' are collinear and A=B. ■

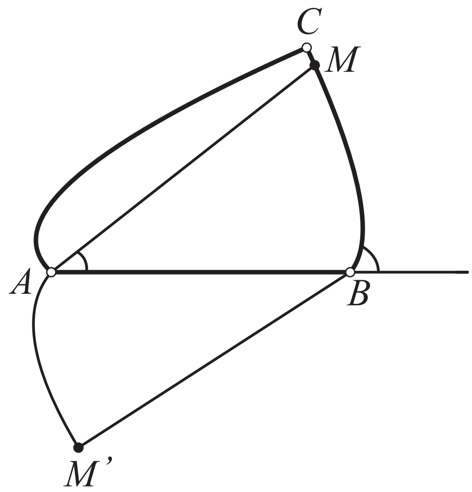


Figure 1.5: Proof: Lemma 1.18

Remark 1.19 *If two lines intersect another line at the same angle, then they cannot intersect each other. $\Rightarrow$ Exist non-intersecting lines in absolute geometry.*

Lemma 1.20 *If, in two triangles $ABC\triangle$ and $A'B'C'\triangle$ the congruences $\overline{AB} \cong \overline{A'B'}$, $ACB\angle \cong A'C'B'\angle$ and either $ABC\angle \cong A'B'C'\angle$ or $BAC\angle \cong B'A'C'\angle$ hold, then the triangles are congruent.*

Proof. Let *e.g.* $ABC\angle \cong A'B'C'\angle$ be true. Let C^* be on $B'C'$ such that $BC^* \cong B'C'$. Then $ABC^*\triangle \cong A'B'C'\triangle$ /Axiom C5/, therefore $AC^*B\angle \cong A'C'B'\angle$ and the previous remark implies that $C' = C^*$. ■

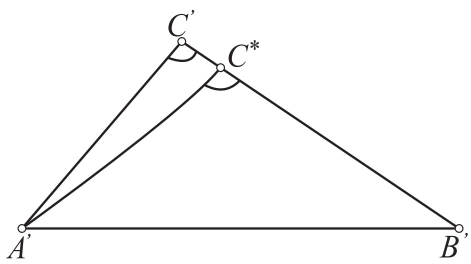


Figure 1.6: Proof: Lemma 1.20

Lemma 1.21 *In any triangle, the greater angle is subtended by the greater side and the greater side subtends the greater angle.*

Proof. Suppose that $\overline{AC} < \overline{BC}$. Let A' be such that $\overline{AC} \cong \overline{A'C}$ and $(CA'B)$. C lies on the bisector of $\overline{AA'}$. Therefore $CAA'\angle \cong CA'A\angle$. Using the external angle lemma, we obtain that $CA'A\angle > CBA\angle$. ■

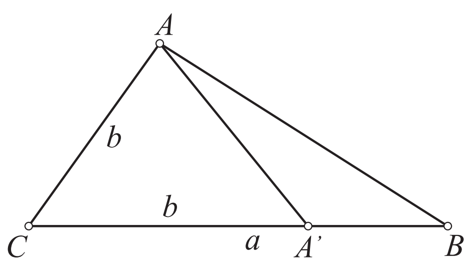


Figure 1.7: Proof: Lemma 1.22

Lemma 1.22 *For any triangle, the sum of the length of any two sides must be greater than the length of the remaining side.*

Proof. Let D be such that (BCD) and $\overline{CD} \cong \overline{AC}$. Then $ACD\triangle$ is isosceles, therefore $CDA\angle \cong CAD\angle$. But $BAD\angle > CAD\angle$ and using the previous lemma, we obtain that $|\overline{BD}| > |\overline{BA}|$. ■

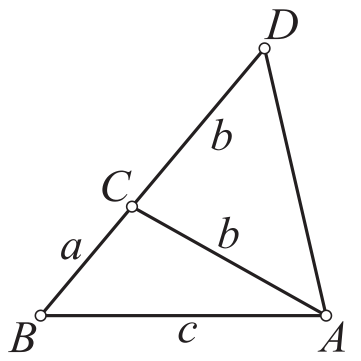


Figure 1.8: Proof: Lemma 1.22

Corollary 1.23 (Polygonal chain inequality) *Let $A_1, A_2, \ldots, A_n$ be a polygonal chain. Then $\sum_{i=1}^{n-1} |\overline{A_iA_{i+1}}| \geq |\overline{A_1A_n}|$.*

Lemma 1.24 (Arm-lemma) *If, in two triangle $ABC\triangle$ and $A'B'C'\triangle$ the congruences $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$ hold and $ABC\angle < A'B'C'\angle$, then $|\overline{AC}| < |\overline{A'C'}|$.*

Proof. We may assume that $A = A'$ and $B = B'$ Due to the lemma 1.17, the line BC intersects AC' and we obtain D.

1. case: (BDC): $AC'C\angle < BC'C\angle \cong BCC'\angle < ACC'\angle$. Therefore using the triangle inequality for the triangle $ACC'\triangle$ we obtain that $|\overline{AC}| < |\overline{A'C'}|$.

2. case: (BCD): In any isosceles triangle, the base angle is smaller then $\frac{\pi}{2}$, since the sum of the two base angle is smaller then the straight angle /External angle lemma/. In our case $BCC'\angle < \frac{\pi}{2} \Rightarrow C'CD\angle > \frac{\pi}{2} \Rightarrow |\overline{C'D}| > |\overline{CD}|$. Using the triangle inequality and, we obtain that: $|\overline{AC}| < |\overline{AD}| + |\overline{DC}| < |\overline{AD}| + |\overline{DC'}| = |\overline{AC'}|$. ■

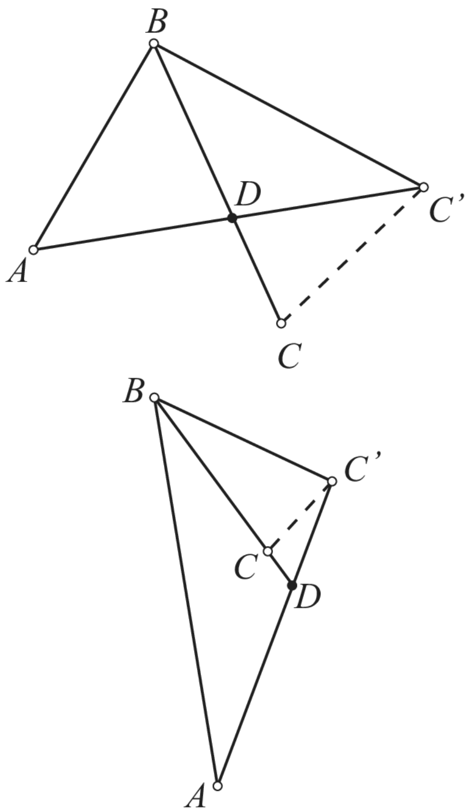


Figure 1.9: Proof: Lemma 1.24

Theorem 1.25 (Legendre I.) *For any triangle, the sum of its internal angles must be less than or equal to π.*

Proof. Indirect The internal angle sum in $ABC\triangle$ is greater than π.
Let $|\overline{A_iA_{i+1}}| = |\overline{AB}|$ be contiguous segments on the line of AB and C_i be such that $A_iC_iA_{i+1}\triangle \cong ACB\triangle$. Due to our indirect assumption $\forall i = 0, 1, \ldots, n-1 : A_iC_iA_{i+1}\angle > C_iA_{i+1}C_{i+1}\angle$. Applying the Arm-lemma: $|\overline{A_0A_1}| = |\overline{A_iA_{i+1}}| > |\overline{C_iC_{i+1}}| = |\overline{C_0C_1}|$. Now, ordering the polygonal chain inequality $|\overline{A_0C_0}| + |\overline{C_0C_1}| + \ldots |\overline{C_{n-1}A_n}| > n|\overline{A_0A_1}|$, we

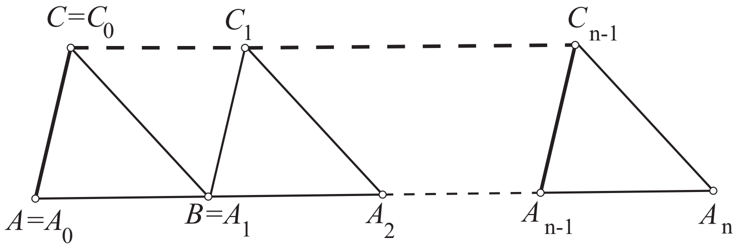


Figure 1.10: Proof: Theorem 1.25

obtain that $n(|\overline{A_0A_1}| - |\overline{C_0C_1}|) < |\overline{A_0C_0}| + |\overline{C_{n-1}A_n}|$ which contradicts the Archimedean axiom. ■

Definition 1.26 *The defect of a triangle is the difference between π and the internal angle sum of the given triangle (additive, non-negative).*

Theorem 1.27 (External angle theorem) *Any external angle of a triangle is larger than or equal to the sum of the other two internal angles.*

Proof. Let D be such that (ABD). In the triangle $ABC\triangle$: $\pi \geq CAB\angle + ABC\angle + ACB\angle$. Therefore, $DBC\angle = \pi - ABC\angle \geq CAB\angle + ACB\angle$. ■

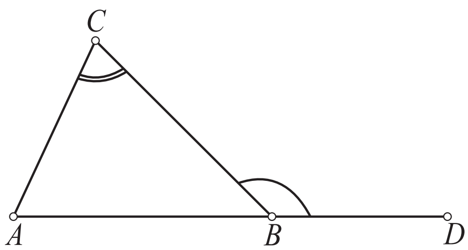


Figure 1.11: Proof: Theorem 1.27

Theorem 1.28 (Legendre II.) *If, the defect of a triangle is 0, then the defect of any triangle is 0.*

Proof. Assuming, that the defect of $ABC\triangle$ is 0, we can divide it into two right angled

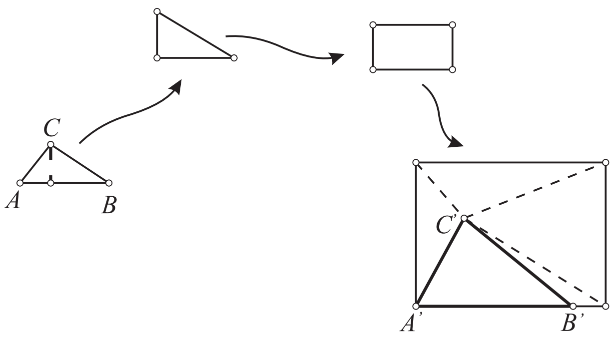


Figure 1.12: Proof: Theorem 1.28

triangle with defect 0. Joining two, we get a rectangle with defect 0. For any triangle $A'B'C'\triangle$, there exists a rectangle, which contains the triangle /Archimedean axiom/. We may divide the rectangle into five triangles. Since the defect is additive and non-negative, the defect of $A'B'C'\triangle$ must be 0. ■

Remark 1.29 *For any non self intersecting polygon, we can define the defect, by dividing it into $n-2$ triangles.*

Theorem 1.30 *The defect of a triangle is* 0 $\iff$ (EUC) *is true.*

Proof. Let a be a line and P be a point not on it. Let PT be perpendicular to a (T is on a) and PK be perpendicular to PT. Then, for the Remark 1.19, $PK \cap a = \emptyset$.
$\Rightarrow$ INDIRECT: Let f be such a ray in $TPK\angle$ that $F \cap a = \emptyset$.
We will find a point D, such that $DPK\angle < fPK\angle$. If, we define the points D_i such that $D_1 = B$ and $|\overline{PD_i}| = |\overline{D_iD_{i+1}}|$, then we get for the angle $DD_iT\angle = \frac{1}{2^{i-1}}DD_1T\angle$. Now, f is a ray in $BPD\triangle$ triangle, therefore Lemma 1.17 guarantees the intersection.
$\Leftarrow$ Let a be the line of AB and $C = P$. Furthermore, let $KCB\angle = CBA\angle$ and $LCA\angle = CAB\angle$. Then $\overrightarrow{CL}$ and $\overrightarrow{CK}$ are such rays, that they do not intersect a $\Rightarrow$ they belong to the same line and the defect of the triangle is 0. ∎

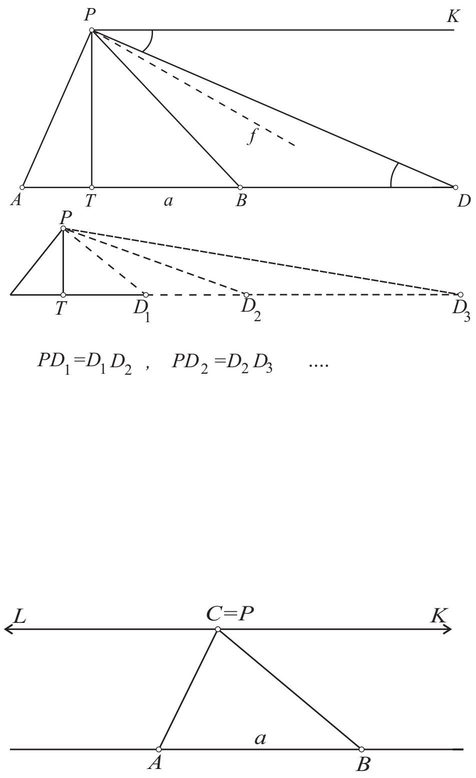


Figure 1.13: Proof: Theorem 1.30

1.3 Orthogonality

Definition 1.31 *Two lines are orthogonal to each other if they divide the plane into four congruent parts.*

Definition 1.32 *A line intersecting a plane is orthogonal to the plane, if it is orthogonal to all the lines, passing through the intersection point.*

Theorem 1.33 *A line intersecting a plane is orthogonal to the plane, if it is orthogonal to two lines, passing through the intersection point.*

Proof. Let a and b be two intersecting lines on a plane and D be their intersection point. Let n be orthogonal to both a and b such that D lies on n. Furthermore, let c be a line in the plane, passing through the point D. There exist always points $A \in a$ and $B \in b$ such that c intersects the segment $\overline{AB} \Rightarrow C$. Let P' be a point on n such that $|\overline{PD}| = |\overline{P'D}|$. Then $PDA\triangle \cong P'DA\triangle$ and $PDB\triangle \cong P'DB\triangle$, since $|\overline{PD}| = |\overline{P'D}|$, they have a side in common and the angles between them is $\frac{\pi}{2}$.

Therefore $|\overline{PA}| = |\overline{P'A}|$ and $|\overline{PB}| = |\overline{P'B}|$. But then $PAB\triangle \cong P'AB\triangle \Rightarrow PAC\angle \cong P'AC\angle \Rightarrow PAC\triangle \cong P'AC\triangle \Rightarrow |\overline{PC}| = |\overline{P'C}| \Rightarrow PDC\triangle \cong P'DC\triangle \Rightarrow PDC\angle \cong P'DC\angle = \frac{\pi}{2}$. ■

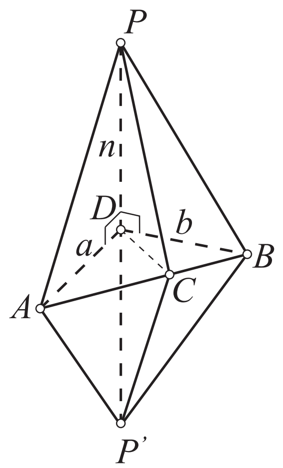


Figure 1.14: Proof: Theorem 1.33

Corollary 1.34 *In the space, the equidistant surface of two point is the orthogonal plane at the midpoint of them.*

Corollary 1.35 *The locus of lines, orthogonal to a given line at a given point of it is a plane, orthogonal to the line at that point.*

Theorem 1.36 *Through a given point A exactly one line can be drawn perpendicularly to a given plane α.*

Proof. *1. case:* A does not lie on α

Let a be an arbitrary line on α and T be a point on a such that $AT \perp a$. First, we draw an orthogonal line to a through T in α, then an orthogonal line to this line through A with foot point D. Let A' be a point on the line of AD such that $|\overline{AD}| = |\overline{A'D}|$ and S be an arbitrary point on a. $A'T \perp a$, since a is perpendicular to the plane of ADT. Then $ADT\triangle \cong A'DT\triangle \Rightarrow |\overline{AT}| = |\overline{A'T}| \Rightarrow ATS\triangle \cong A'TS\triangle \Rightarrow |\overline{AS}| = |\overline{A'S}| \Rightarrow ADS\triangle \cong A'DS\triangle \Rightarrow AA' \perp DS$

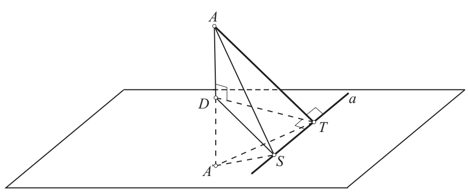


Figure 1.15: Proof: Theorem 1.36

2. case: A lies on α

Our only solution will be the intersection line of the planes, perpendicular to two arbitrary lines, passing through the point A. ■

Theorem 1.37 (of three perpendiculars) *Let a be a line on a given plane and A be a point not lying in the plane. The the foot points T and D of the perpendiculars from A to a and α respectively form a perpendicular line to a.*

Proof. Let S and R be equidistant points on a from T. Then $TAS\triangle \cong TAR\triangle$, therefore $|\overline{AS}| = |\overline{AR}|$. But this implies that $ADS\triangle \cong ADR\triangle \Rightarrow |\overline{DS}| = |\overline{DR}|$. Now $DSR\triangle$ is an isosceles triangle and $DT \perp SR$. ■

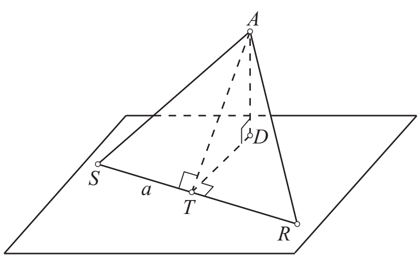


Figure 1.16: Proof: Theorem 1.37

Definition 1.38 *The orthogonal projection of a point of the space for a given plane is the common point of the plane and the line, perpendicular to the plane through the point. The orthogonal projection of a point set is the set of the projected points.*

Definition 1.39 *A mapping is a collineation if the image of collinear points are collinear themselves.*

Theorem 1.40 *The orthogonal projection is a collineation and it preserves incidence.*

Proof. Let A and B be two points on a line and their projection be A' and B' respectively. Then A, A', B and B' are coplanar.
INDIRECT: Let the line e be in the plane, perpendicular to $A'B'$ through A' and F be the midpoint of the segment $\overline{A'B'}$. Reflecting the lines AA', $A'B'$ and e respected to F, we obtain the lines $B'C$, $A'B'$ and f respectively. Since the orthogonal line to the plane through a given point is unique, B is on the line $B'C$.
Now, let A, B and C be collinear points. Then the points A, A', B, B' and A, A', C, C' are coplanar $\Rightarrow$ the two plane are the same $\Rightarrow$ A', B' and C' are collinear points. ■

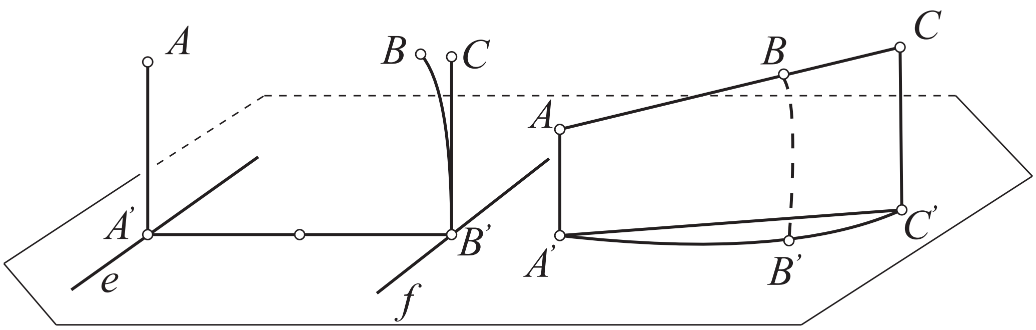


Figure 1.17: Proof: Theorem 1.40

Definition 1.41 *Two planes are orthogonal to each other if one of them contains a line, perpendicular to the other.*

Theorem 1.42 *For every plane and line, there always exists a plane, which is orthogonal to the given plane and contains the given line.*

Proof. If the orthogonal projection of the line is a single point, then it is orthogonal to the plane, and any plane which contains the line will be appropriate.
If the orthogonal projection of the line is a line does not coincide with the original line, then they determine this orthogonal plane.
If the orthogonal projection of the line is a line coincides with the original line, then construct a perpendicular line to the given line in the given plane at any point if it. The orthogonal plane to the constructed line through this point will satisfy the conditions. ■

2 Hyperbolic geometry

2.1 Parallel lines on the hyperbolic plane

Let TA be perpendicular to a such that T lies on a. Furthermore let $\overrightarrow{AK}$ be perpendicular to AT $\Rightarrow AK \cap a = \emptyset$. Consider the rays emanating from A in $TAK\angle$. They intersect the segment $\overline{TK}$ for Lemma 1.17. Assign a ray to the set $\mathcal{A}$ if it intersects a and to $\mathcal{B}$ if not. Then, apply the Dedekind axiom to these sets $\Rightarrow$ *limiting parallels*.

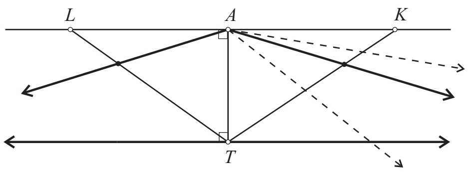


Figure 2.1: Limiting parallels

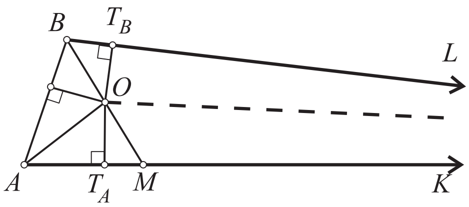
Definition 2.1 *$\overrightarrow{AC} \parallel \overrightarrow{TR}$, if $\overrightarrow{AC}$ is the limiting parallel to $\overrightarrow{TR}$. If $D \in \mathcal{A}$, then $\overrightarrow{AD}$ called intersecting line of $\overrightarrow{TR}$. If $E \in \mathcal{B}$, then $\overrightarrow{AE}$ called ultraparallel line to $\overrightarrow{TR}$.*

Lemma 2.2 *If $\overrightarrow{AB} \parallel \overrightarrow{CD}$, then for any point C^* on the line of CD: $\overrightarrow{AB} \parallel \overrightarrow{C^*D}$.*

Proof. One can distinguish 2 cases, both can be proved by the Pasch axiom. ■

Theorem 2.3 *The parallelism of rays is symmetric and transitive.*

Proof. Symmetry: Let $\overrightarrow{AK} \parallel \overrightarrow{BL}$ be such that L and K are points "far enough". Draw the angle bisector of $BAK\angle$ and $ABL\angle$, they intersect each other in O. Let T_A and T_B be points on $\overrightarrow{AK}$ and $\overrightarrow{BL}$ respectively, such that $OT_A \perp AK$ and $OT_B \perp BL$. Then $\overrightarrow{T_AK} \parallel \overrightarrow{T_BL}$ for the previous lemma. But then $\overrightarrow{T_BL} \parallel \overrightarrow{T_AK}$, since it is symmetric to the angle bisector of $T_BOT_A\angle$.



Transitivity (Only in the plane and only for one case): $\overrightarrow{AK} \parallel \overrightarrow{BL} \wedge \overrightarrow{BL} \parallel \overrightarrow{CM} \overset{?}{\Rightarrow} \overrightarrow{AK} \parallel \overrightarrow{CM}$

1) If $\overrightarrow{CM} \cap \overrightarrow{AK} \neq \emptyset \underbrace{\Rightarrow}_{\text{Pasch}} \overrightarrow{CM} \cap \overrightarrow{BL} \neq \emptyset$.

2) If f is a ray, emanating from C in $BCM\angle \Rightarrow f \cap \overrightarrow{BL} \neq \emptyset \Rightarrow B'$ and $\overrightarrow{B'L} \parallel \overrightarrow{AK} \Rightarrow f$ also intersects $\overrightarrow{AK}$ ■

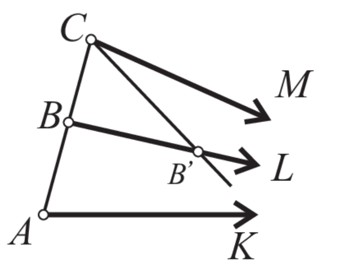


Figure 2.2: Proof: Theorem 2.3

Definition 2.4 *Two line is parallel to each other if they have parallel rays.*

2.1.1 Perpendicular transverse of ultraparallel lines

Theorem 2.5 (Bolyai) *Every pair of ultraparallel lines has a unique line, perpendicular to both lines (perpendicular transverse).*

Proof. We use without proof, that the angle, formed by a given line and the intersecting rays, emanating from a point, not on the line, changes continuously. Let the $\overrightarrow{AK}$ and $\overrightarrow{BL}$ be two rays on two ultraparallel lines, such that $BAK\angle = \frac{\pi}{2}$ and $ABL\angle < \frac{\pi}{2}$. Then exists a point M on the ray $\overrightarrow{AK}$ such that $AMB\angle \cong MBL\angle$. Let F be the midpoint of the segment $\overline{BM}$. Let b be a perpendicular line to AK such that F lies on b. Then $TFM\triangle \cong SFB\triangle \Rightarrow BSF\angle = \frac{\pi}{2}$. ■

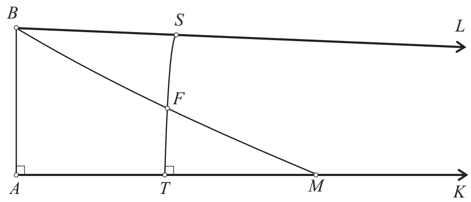


Figure 2.3: Proof: Theorem 2.5

Definition 2.6 *A quadrilateral, with two equal sides perpendicular to the same (base) side called* Saccheri quadrilateral.

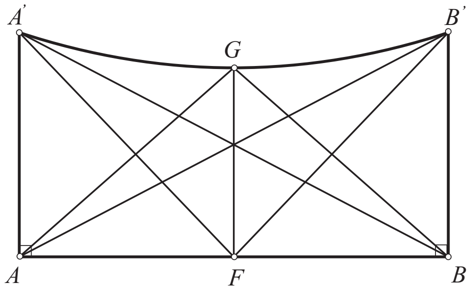


Figure 2.4: Proof: Lemma 2.7

Lemma 2.7 *1.* $AA'B'\angle \cong BB'A'\angle < \frac{\pi}{2}$

2. $A'G \cong GB' \wedge AF \cong FB \Rightarrow FG \perp AB \wedge FG \perp A'B'$

3. $|\overline{A'B'}| > |\overline{AB}|$

Proof. $BAB'\triangle \cong ABA'\triangle$, because $\overline{AB}$ is common, $\overline{AA'} \cong \overline{BB'}$ and $A'AB\angle \cong B'BA\angle = \frac{\pi}{2} \Rightarrow \overline{AB'} \cong \overline{A'B} \wedge B'AB\angle \cong A'BA\angle \Rightarrow A'AB'\angle \cong B'BA'\angle \Rightarrow AA'B'\triangle \cong BB'A'\triangle \Rightarrow$ *1.*

Similarly, $AA'F\triangle \cong BB'F\triangle$ and therefore $A'FG\triangle \cong B'FG\triangle \Rightarrow A'GF\angle \cong FGB'\angle = \frac{\pi}{2} \Rightarrow$ *2.*

Consider now the triangles $A'AB\triangle$ and $B'BA'\triangle$. Since $B'BA'\angle + A'BA\angle = \frac{\pi}{2}$ and $AA'B\angle + A'BA\angle < \frac{\pi}{2} \Rightarrow AA'B\angle < B'BA\angle$. Using the Arm-lemma, we obtain *3.* ■

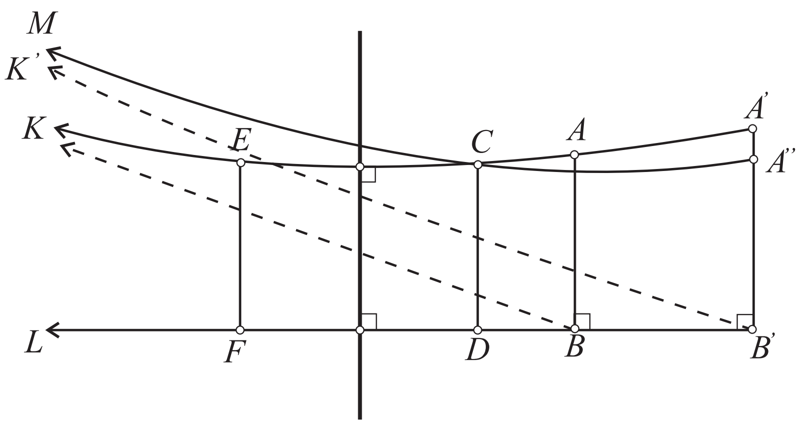


Figure 2.5: Hilbert construction

Construction. (Hilbert): Suppose that $AB \perp r$, $KAB\angle < \frac{\pi}{2}$, (KAA'), $A'B' \perp r$ and $|\overline{A''B}| = |\overline{AB}|$. Then $(A'A''B')$, because $ABB'A''$ is a Saccheri quadrilateral $\Rightarrow BAA''\angle < \frac{\pi}{2} < BAA'\angle$. Let s' be a line such that $KAB\angle \cong K'A''B'\angle$. Then $s \cap s' \neq \emptyset$, since $\overrightarrow{BL} \cap \overrightarrow{AK} = \emptyset \Rightarrow \overrightarrow{BK} \parallel \overrightarrow{AK}$ and $\overrightarrow{B'K'} \parallel \overrightarrow{A''K'}$, furthermore $LBK\angle \cong LB'K'\angle$. Then $\overrightarrow{BK}$ and $\overrightarrow{B'K'}$ are ultraparallel rays $\Rightarrow s \cap B'K' \neq \emptyset \Rightarrow P$. Using the Pasch axiom in $BAA'\triangle$ with $B'K'$ we obtain that $B'K'$ intersects either $\overline{AA'}$ or $\overline{AB}$. If it is $\overline{AA'}$, then we are done, otherwise in $KBA\triangle$ we use the Pasch axiom again, but $B'K'$ cannot intersect $BK \Rightarrow$ it must intersect AK. Let C be $s \cap s'$ and $D \in r$ b such that $CD \perp r$. Construct a point E on s so that $\overline{AE} \cong \overline{A''C}$. Then let $F \in r$ be such that $EF \perp r$. Then $EFCD$ is a Sacchery quadrilateral $\Rightarrow$ the symmetry axis is good. ■

2.2 Line pencils and cycles

Definition 2.8 *The lines pencils are set of lines:*

- *passing through a given (finite) point*
- *parallel to a given line*
- *perpendicular to a given line.*

Definition 2.9 *A cycle is the orbit of a point, reflecting it to a given line pencil:*

- *cycle (finite point)*
- *horocycle/paracycle (parallel)*
- *hypercycle/equidistant line (perpendicular).*

Theorem 2.10 *If, a cycle has three collinear points, then it is a line.*

Proof. Let $P'$and P'' be the reflections of the point P respected to two elements of the line pencil $\Rightarrow$ they are the perpendicular bisector of the the segments $\overline{PP'}$ and $\overline{PP''}$. We have the perpendicular transverse of these lines $\Rightarrow$ ultraparallel lines $\Rightarrow$ the points lie on a hypercycle $\Rightarrow$ the point lie on the base line. ■

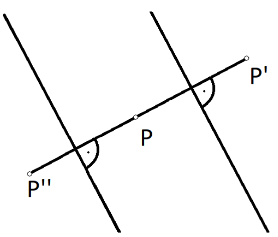


Figure 2.6: Proof: Theorem 2.10

Definition 2.11 *Let A and B be two points on two parallel rays. We say that A and B are corresponding points if the segment $\overline{AB}$ forms equal angles with the rays.*

Lemma 2.12 Circle*: Locus of point, that are equidistant from a given point.*
Paracycle*: Locus of points, that are corresponding to a given point, respected to a given parallel line pencil.*
Hypercycle*: Locus of points, that are equidistant from a given point.*

2.3 Models of the hyperbolic geometry

1. CAYLEY-KLEIN DISK MODEL
   Points: Interior of the unit disk
   Lines: Chords
   Axioms: I-IV trivial, since it is part of the Euclidean plane
   Parallels: Chords, sharing an endpoint (boundary point)
   Perpendiculars: $f \perp g$ if and only if f goes through the intersection point of the tangents of g.

2. POINCARÉ DISK MODEL /conformal disk model/
   Points: Interior of the unit disk
   Lines: Diameters and circular arcs, perpendicular to the model circle
   Axioms: I-IV trivial, since it is part of the Euclidean plane
   Parallels: Circular arcs and diameters, sharing an endpoint (boundary point)

3. POINCARÉ HALF-PLANE MODEL
   Points: Upper half-plane
   Lines: Rays and circular arcs, perpendicular to the model circle
   Axioms: I-IV trivial, since it is part of the Euclidean plane
   Parallels: Circular arcs and rays, sharing an endpoint (boundary point)

Remark 2.13 *Both the Poincaré disk and half-plane model are conformal models: The angle of lines seems real size in these models.*

2.3.1 Orthogonality in the Cayley-Klein model

Lemma 2.14 *If, the length of the tangents, drawn from an external point to two intersecting circles, are equal, then the point lies on the common secant of the circles.*

Proof. Let K be the external point and $|\overline{KT}| = |\overline{KR}|$. The ray, emanating from K through one of the intersection point A intersects the circles in points B and C. Using the intersecting secant theorem: $|\overline{KA}||\overline{KB}| = |\overline{KT}|^2 = |\overline{KR}|^2 = |\overline{KA}||\overline{KC}| \Rightarrow |\overline{KB}| = |\overline{KC}| \Rightarrow B = C$. ■

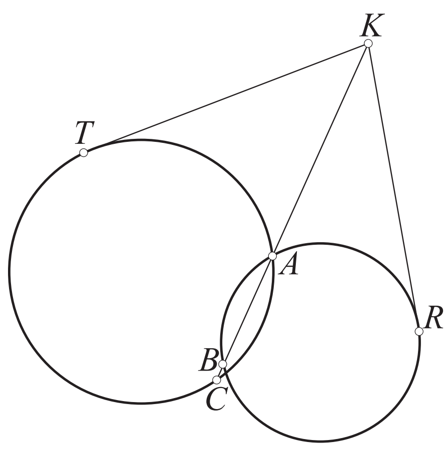


Figure 2.7: Proof: Lemma 2.14

Definition 2.15 *Two chords of a said to be conjugate to each other if the intersection point of the tangent, drawn to the circle at the endpoints of one of the chords lies on the line of the other chord.*

Lemma 2.16 *Two lines are orthogonal to each other in the Cayley-Klein model if and only if the representing chords are conjugate to each other.*

Proof. Let KL and $K'L'$ be orthogonal lines in the Poincaré disk model, intersecting each other at the point P. Then $|\overline{OP}| = |\overline{OK'}| = |\overline{OL'}|$, therefore O must lie on the common secant of the circles, determined by the points $KK'LL'$ and KPL, for the previous lemma. ■

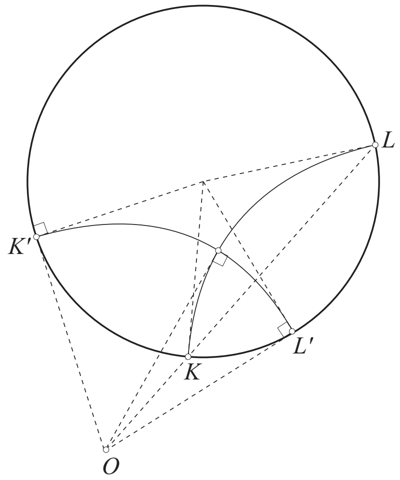


Figure 2.8: Proof: Lemma 2.16

2.3.2 Stereographic projection

1. Defined as the projection of the sphere from the North/South pole onto the equatorial plane.

2. This is a bijective mapping between the unit sphere and the plane (usualy $S^2 \to \mathbb{R}^2/\mathbb{C}$).

3. It is equivalent with the projection from the North pole onto the tangent plane of the South pole.

Theorem 2.17 *The stereographic projection preserves circles and it is conformal.*

Proof. Let P and Q be two points on the surface of the sphere, and their projections are P' and Q'. Let the intersection of the line PQ and $P'Q'$ be N, and S be the intersection of the line PQ and the tangent plane of the center of the projection O. First, we prove the $PQQ'P'$ is a cyclic quadrilateral. OS is the tangent line of the triangle $OQP\triangle$. Because of the inscribed angle theorem, the supplementary angle of $QOS\angle$ is equal to $QPO\angle$. Since $OS \parallel Q'N$, the supplementary angle of $QOS\angle$ is also equal to $OQ'P'\angle$. Therefore $|\overline{NP}||\overline{NQ}| = |\overline{NP'}||\overline{NQ'}|$.
Now let $|\overline{PQ}|$ be a chord of a circle on the sphere. Then the product $|\overline{NP}||\overline{NQ}|$ is constant for every $|\overline{PQ}|$ chord and equal to the product $|\overline{NP'}||\overline{NQ'}|$. This is true for every point N, which lies on the intersection line of the projection plane and the plane, that contains the circle. The intersection of the equatorial plane with the elliptical cone of the circle and O is also a circle.

Conformal: Let P be a vertex of an angle with tangent lines t_1 and t_2. Let k_i be a spherical circle in the plane of t_i and O such that P and O lie on k_i. Then the tangent of these circles s_i meet at the same angle and they lie a parallel plane to the tangent plane of O. Therefore t_1' and t_2' also meet at the same angle. ■

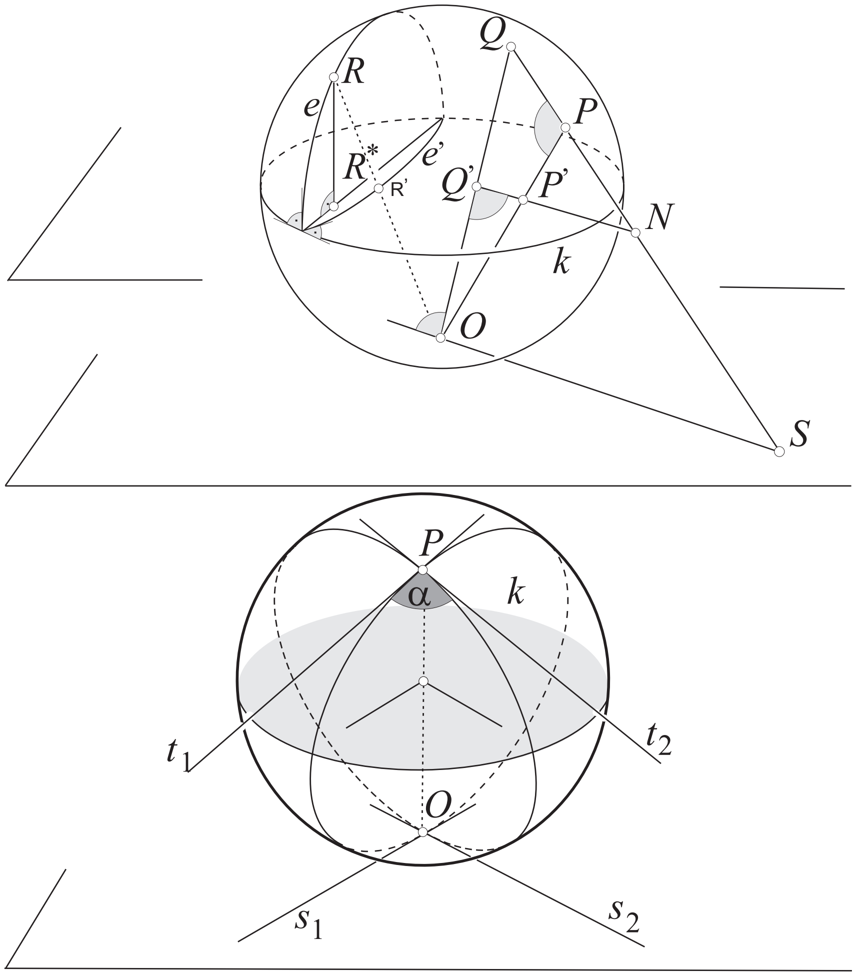


Figure 2.9: Proof: Theorem 2.17

Theorem 2.18 *Consider the unit disk with the Poincaré structure. Then, the composition of the inverse stereographic projection with orthogonal projection back to the plane of the circle is a bijection on the disk, the ideal points are fix points and it results in the Cayley-Klein model.*

Proof. Only the lines are different in the two models. Since a line in the Poincaré disk model is a circle, orthogonal to the great circle, the inverse stereographic projected image is also a circle, orthogonal to the main circle. The plane of it is perpendicular to the base plane ⇒ the orthogonal projection of the circular arc is a line segment, connecting the endpoints of the Poincaré line (circle). ■

2.3.3 Inversion

Definition 2.19 *Let O be the center of a circle/sphere with radius r. The inverse image of a point $P(\neq O)$ is the point P' if P' lies on the ray $\overrightarrow{OP}$ and $|\overline{OP}||\overline{OP'}| = r^2$. The mapping, that assigns the inverse image to every point of the plane/space, is called inversion.*

Theorem 2.20 *The image of a circle or a line by inversion is either a circle or a line. Inversion is a conformal mapping.*

Proof.

- The base circle of the inversin is fixed point by point.
- Every line, through the center of the inversion is also invariant (inside $\longleftrightarrow$ outside).
- The image of a line, which does not contain O, is a circle, through O:
  Let P be the foot point of the orthogonal line to the given line, through O and Q be an arbitrary point on the given line. Let P' and Q' be the inverse image of P and Q respectively. Then $|\overline{OP}||\overline{OP'}| = |\overline{OQ}||\overline{OQ'}| \Rightarrow \frac{|\overline{OP}|}{|\overline{OQ}|} = \frac{|\overline{OQ'}|}{|\overline{OP'}|} \Rightarrow$ $OP'Q'\triangle \sim OQP\triangle \Rightarrow P'Q'O\angle = \frac{\pi}{2} \Rightarrow Q'$ lies on the Thales circle above the segment $\overline{OP'}$ as diameter.
- The image of a circle, which contains O is a line, which does not contain O.
- The image of a circle, which does not contain O, is also a circle, which does not contain O:
  Let PQ be a secant line through O. Then the product of $|\overline{OP}||\overline{OQ}| = p$ is a constant independent from the secant. Applying a scaling from O by ratio $1 : p$, we obtain the l^* circle with center K^*. The secant intersects l^* and the image of Q is Q'. Then $|\overline{OQ^*}||\overline{OP}| = \frac{|\overline{OQ}|}{p}|\overline{OP}| = 1$

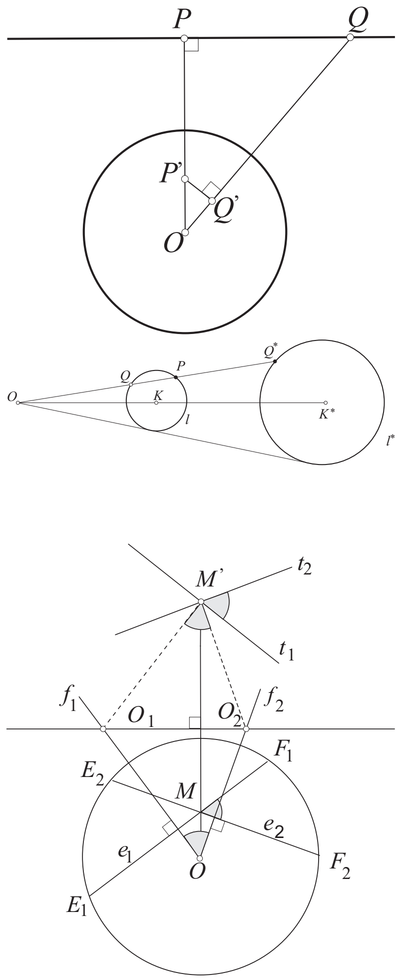


Figure 2.10: Proof: Theorem 2.20

Conformal: Let e_1 and e_2 be the tangents to the curves, their intersection be M and the image of M be M'. Let f_i be perpendicular lines to e_i through O, and O_i be the intersection points of f_i by the perpendicular bisector of $\overline{OM'}$. Then the images of e_i by the inversion are circles with center O_i through O. Then $t_i \perp O_iM'$, therefore $t_1t_2\angle \cong O_1M'O_2\angle \cong O_1OO_2\angle \cong e_1e_2\angle$. ■

Remark 2.21 *Applying the inversion to the Poincaré disk/sphere model at a boundary point, we obtain the Poincaré half-plane/half-space model. We also have the advantage, that this point is arbitrary, therefore we can choose the representative of one line/plane in the model to be a ray/half plane. Because the Poincaré disk/sphere model is a conformal model and inversion is a conformal mapping, the Poincaré half-plane/half-space model is also a conformal model.*

2.4 Distance and angle of space elements

2.4.1 Mutual position of space elements ($\mathbf{E}^3$)

The parallelism of lines is an equivalence relation (parallel line pencil). We assign an ideal point for every equivalence class. Two line intersect each other at this ideal point, if they are parallel in the Euclidean space. The union of ideal points forms an ideal plane.

Definition 2.22 *The projective space* $\mathbf{P}\mathbb{R}^3$ *is the union of the Euclidean space and the ideal elements.*

1. Line-Line
   - intersecting: if their intersection is not ideal
   - parallel: if they are not intersecting in $\mathbf{E}^3$ but in $\mathbf{P}\mathbb{R}^3$
   - skew: if they are not intersecting in $\mathbf{P}\mathbb{R}^3$
2. Line-Plane/Plane-Plane
   - intersecting
   - parallel

2.4.2 Mutual position of space elements ($\mathbf{H}^3$)

1. Line-Line
   - intersecting
   - parallel
   - ultraparallel
   - skew (not intersecting in $\mathbf{P}\mathbb{R}^3$)
2. Line-Plane
   - intersecting
   - parallel
   - skew (they intersect each other either outside of the model or only in $\mathbf{P}\mathbb{R}^3$)

3. Plane-Plane
    - intersecting
    - parallel
    - ultraparallel

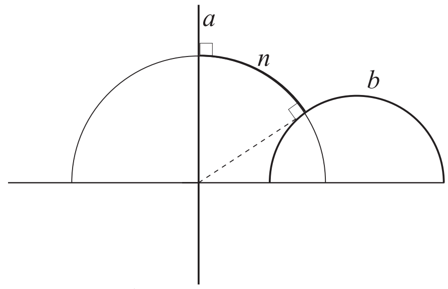
2.4.3 Perpendicular transverse of non-intersecting hyperbolic space elements

Theorem 2.23 *In the hyperbolic space, there exits a unique perpendicular transverse line for two skew lines, for two ultraparallel lines/planes.*

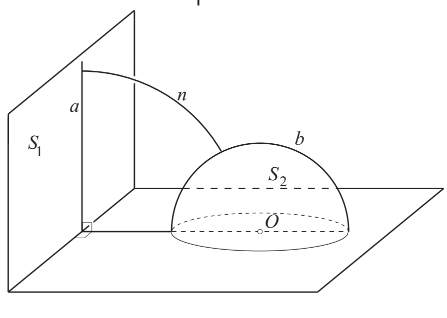
Proof. We use the Poincaré half-plane/half-space model.

1. *Ultraparallel lines (planar case)*

2. *Ultraparallel line-plane*
   Perpendicular line in the boundary plane to the line through the center of the plane (sphere).



3. *Ultraparallel planes*
   We draw a perpendicular line to the intersection of the plane and the boundary plane through the center of the sphere. We apply the previous case for the sphere and the line, perpendicular to the boundary plane in the given plane through the footpoint of the previous line.



4. *Skew lines*
   Let k be a half-sphere, perpendicular to the boundary plane, through $A, B, C \Rightarrow b \in k$. FD is the perpendicular bisector of $\overline{BC} \Rightarrow S = AD \cap BC$. Let N be a point on b uch that $N^* = S$. Then n will be a circle perpendicular both a and b with center A and radius $|\overline{AN}|$. The perpendicularity to a is obvious. Let t_b and t_n be the tangents of b and n in N. t_b is in the plane of $BNC \Rightarrow t_b \perp FD, NF \Rightarrow t_b \perp ND$. In the triangle $AND\triangle$ AD is the diameter of the circumscribed circle $\Rightarrow AND\angle = \frac{\pi}{2} \Rightarrow t_n = ND \Rightarrow t_n \perp t_b \Rightarrow n \perp b$. ∎

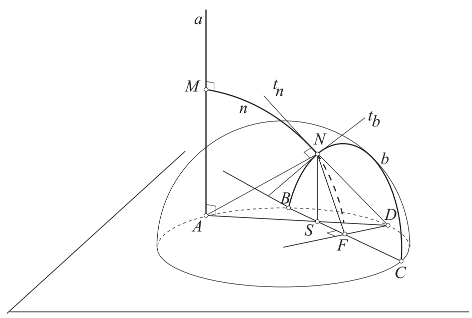


Figure 2.11: Proof: Theorem 2.23

2.4.4 Cross-ratio

Definition 2.24 *Let z_1, z_2, z_3, z_4 be complex numbers. Then the cross-ratio:*

$$(z_1, z_2, z_3, z_4) := \frac{z_3 - z_1}{z_2 - z_3} : \frac{z_4 - z_1}{z_2 - z_4}.$$

Theorem 2.25 *The cross-ratio of 4 distinct points is real, if and only if they all lie either on a circle or on a line.*

Proof. We use the polar decomposition: $z = re^{i\Theta}$

$$\frac{z_3 - z_1}{z_2 - z_3} : \frac{z_4 - z_1}{z_2 - z_4} = \frac{z_1 - z_3}{z_2 - z_3} \cdot \frac{z_4 - z_2}{z_4 - z_1} =$$

$$= \left|\frac{z_1 - z_3}{z_2 - z_3}\right| e^{i\Theta_3} \left|\frac{z_1 - z_4}{z_2 - z_4}\right| e^{i\Theta_4} = Re^{i(\Theta_3 + \Theta_4)}$$

where R is a real number. $e^{i(\Theta_3+\Theta_4)} \in \mathbb{R} \Leftrightarrow \Theta_3 + \Theta_4 = k\pi (k \in \mathbb{Z})$. If $\Theta_3 + \Theta_4 = 2k\pi$ then they lie on a circle, otherwise they lie on a line. ■

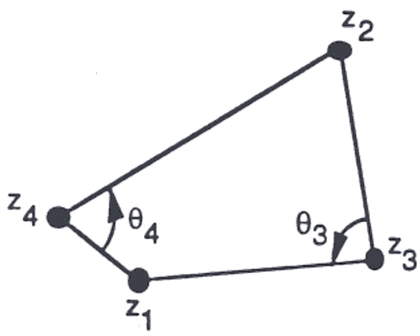


Figure 2.12: Proof: Theorem 2.25

2.4.5 Distances in the hyperbolic space

Definition 2.26 *The hyperbolic distance in the Poincaré disk model: $d(X,Y) = \log(X,Y,U,V)$, where U and V are the endpoints of the line, determined by X and Y such that X,Y,U,V is the cyclic order of the points on the representing circle.*

– point-point: definition
– point-line: distance of the given point and the footpoint of the orthogonal line to the given line through the given point
– point-plane: distance of the given point and the footpoint of the orthogonal line to the given plane through the given point
– line-line: length of the perpendicular transverse line segment
– line-plane: length of the perpendicular transverse line segment
– plane-plane: length of the perpendicular transverse line segment

2.4.6 Angle and distance in the Cayley-Klein model

– Angle: We follow the mapping from the Poincaré structure to the Cayley-Klein structure:

$\alpha = (u,v)\angle = (u',v')\angle = VP'U\angle$

$|\overline{UV}|^2 = |\overline{P'U}|^2 + |\overline{P'V}|^2 - 2|\overline{P'U}||\overline{P'V}|\cos(\alpha)$

$|\overline{P'U}|^2 = r_1^2 = u_1^2 + u_2^2 - 1,\ |\overline{P'V}|^2 = r_2^2 = v_1^2 + v_2^2 - 1$

$|\overline{UV}|^2 = (u_1 - v_1)^2 + (u_2 - v2)^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2u_1v_1 - 2u_2v_2$

$$\frac{r_1^2 + r_2^2 - |\overline{UV}|^2}{2r_1r_2} = \frac{-2 + 2u_1v_1 + 2u_2v_2}{2\sqrt{(-1 + u_1^2 + u_2^2)(-1 + v_1^2 + v_2^2)}} =$$

$$\cos(\alpha) = \frac{-1 + u_1v_1 + u_2v_2}{\sqrt{(-1 + u_1^2 + u_2^2)(-1 + v_1^2 + v_2^2)}} \qquad \text{(C-K}\angle\text{)}$$

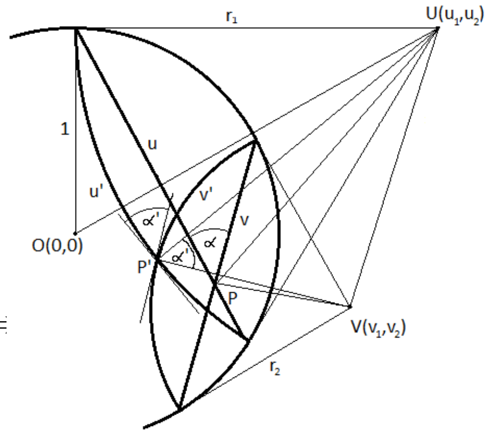


Figure 2.13: Angle in the Cayley-Klein model

– Distance:

$$\cosh(d(\mathbf{x},\mathbf{y})) = \frac{-1 + x_1y_1 + x_2y_2}{\sqrt{(-1 + x_1^2 + x_2^2)(-1 + y_1^2 + y_2^2)}}$$

2.4.7 Hyperboloid model

We use the $\mathbf{V^3}$ real vector space with the standard $\{e_1, e_2, e_3\}$ base. We introduce the symmetric bilinear form: $\langle e_1, e_1\rangle = 1,\ \langle e_2, e_2\rangle = 1,\ \langle e_3, e_3\rangle = -1,\ \langle e_i, e_j\rangle = 0,\ (i \neq j)$. If $\mathbf{x}, \mathbf{y} \in \mathbf{V}^3$, then $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + x_2y_2 - x_3y_3$, and

$$\langle \mathbf{x}, \mathbf{x}\rangle \begin{cases} < 0 & \text{time} - \text{like} \\ = 0 & \text{light} - \text{like} \quad \Rightarrow \\ > 0 & \text{space} - \text{like} \end{cases}$$

$$\langle \mathbf{x}, \mathbf{x}\rangle = r^2 \begin{cases} \text{two} - \text{sheeted hyperboloid} & r \in \mathbb{C} \\ \text{cone} & r = 0 \\ \text{one} - \text{sheeted hyperboloid} & r \in \mathbb{R} \end{cases}$$

We define an equivalence relation: $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \exists c \in \mathbb{R} \setminus \{0\} : \mathbf{y} = c\mathbf{x} \Rightarrow$ representing elements: $\mathbf{x}' \sim (x_1, x_2, 1)$.

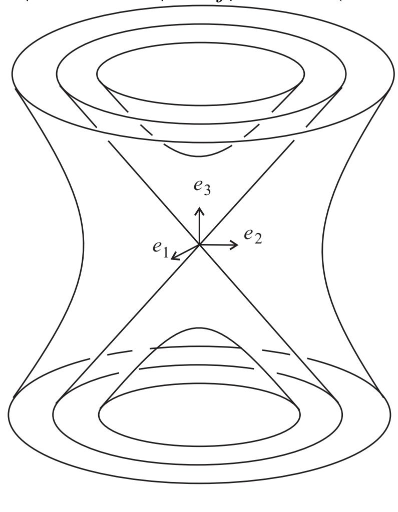


Figure 2.14: Hyperboloids

We assign the time-like vectors to the points of the Cayley-Klein model of the hyperbolic geometry by this equivalence relation.

We define the distance of two points by the $d(X,Y) = \frac{1}{2}\log(X,Y,U_1,U_2)$, where U_1 and U_2 are the boundary points on the line of X and Y such that X, Y, U_1 and U_2 are in cyclic order. Then

$$e^{2d(X,Y)} = (X,Y,U_1,U_2) = \frac{\beta_1}{\alpha_1} : \frac{\beta_2}{\alpha_2} \text{ where } \mathbf{u_1} = \alpha_1\mathbf{x} + \beta_1\mathbf{y}, \text{ and } \mathbf{u_2} = \alpha_2\mathbf{x} + \beta_2\mathbf{y}.$$

$\langle u_i, u_i \rangle = \alpha_i^2 \langle \mathbf{x}, \mathbf{x} \rangle + 2\alpha_i \beta_i \langle \mathbf{x}, \mathbf{x} \rangle + \beta_i^2 \langle \mathbf{y}, \mathbf{y} \rangle = 0 \Rightarrow$
$\langle \mathbf{x}, \mathbf{x} \rangle + 2\frac{\beta_i}{\alpha_i} \langle \mathbf{x}, \mathbf{x} \rangle + \left(\frac{\beta_i}{\alpha_i}\right)^2 \langle \mathbf{y}, \mathbf{y} \rangle = 0$

$$\left(\frac{\beta_i}{\alpha_i}\right)_{1,2} = \frac{-2\langle \mathbf{x}, \mathbf{y} \rangle \pm \sqrt{4\langle \mathbf{x}, \mathbf{y} \rangle^2 - 4\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}{2\langle \mathbf{y}, \mathbf{y} \rangle} \Rightarrow$$

$$e^{2d} = (\mathbf{x}, \mathbf{y}, \mathbf{u}_1, \mathbf{u}_2) = \underbrace{\frac{\langle \mathbf{x}, \mathbf{y} \rangle - \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}{\langle \mathbf{x}, \mathbf{y} \rangle + \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}}_{a}$$

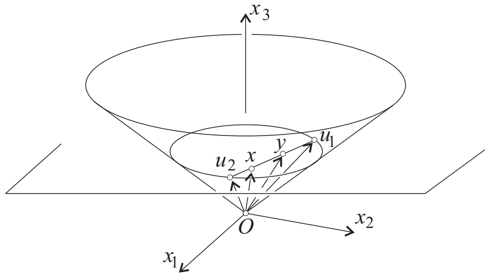


Figure 2.15: Distance in the hyperboloid model

$$\cosh(d(\mathbf{x}, \mathbf{y})) = \frac{e^d + e^{-d}}{2} = \frac{\sqrt{a} + \frac{1}{\sqrt{a}}}{2}, \text{ but } \sqrt{a} + \frac{1}{\sqrt{a}} = \sqrt{(\sqrt{a} + \frac{1}{\sqrt{a}})^2} = \sqrt{a + \frac{1}{a} + 2}$$

$$a + \frac{1}{a} + 2 = \frac{\langle \mathbf{x}, \mathbf{y} \rangle - \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}{\langle \mathbf{x}, \mathbf{y} \rangle + \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}} + \frac{\langle \mathbf{x}, \mathbf{y} \rangle + \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}{\langle \mathbf{x}, \mathbf{y} \rangle - \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}} + 2 =$$

$$= \frac{\left(\langle \mathbf{x}, \mathbf{y} \rangle - \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}\right)^2 + \left(\langle \mathbf{x}, \mathbf{y} \rangle + \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}\right)^2}{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \left(\sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}\right)^2} + 2 =$$

$$= \frac{2\langle \mathbf{x}, \mathbf{y} \rangle^2 + 2\left(\sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}\right)^2}{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle} + 2 = \frac{4\langle \mathbf{x}, \mathbf{y} \rangle^2 - 2\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle} + 2 =$$

$$= \frac{4\langle \mathbf{x}, \mathbf{y} \rangle^2}{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle} \Rightarrow \cosh\left(\frac{1}{2}\log(X, Y, U_1, U_2)\right) = \cosh(d(X, Y)) = \frac{-\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}.$$

2.4.8 Pole-polar relation

To represent a line u in the hyperboloid model, we consider a plane, orthogonal to $\mathbf{u}(u_1, u_2, 1)$ through the origin: $xu_1 + yu_2 + z = 0$. Intersecting it with the plane $z = 1$, we get a line with the equation $xu_1 + yu_2 + 1 = 0$. If $u_1^2 + u_2^2 > 1$, then it can be assigned as a proper line in the Cayley-Klein model, and $X(\mathbf{x})$ lies on it, if $xu_1 + yu_2 + 1 = 0$. The pole $V(v_1, v_2, 1)$ of this line u is the intersection of the tangents to the boundary point of u. This is also the center of the circle, with determines u in the Poincaré model. To get the radius r of this circle, we apply the Pythagorean theorem: $r^2 = a^2 - 1 = v_1^2 + v_2^2 - 1$.

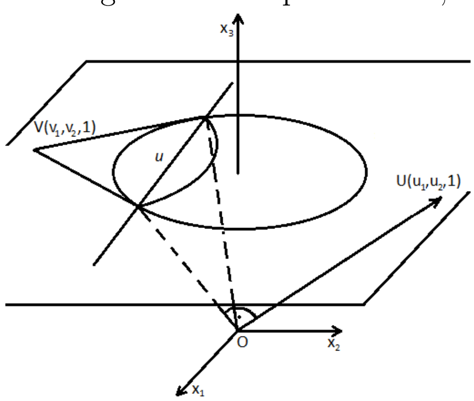


Figure 2.16: Pole-polar

Lemma 2.27 *A line* $\mathbf{u}(u_1, u_2, 1)$ *has the pole* $(-u_1, -u_2, 1)$ *and the point* $P(x, y, 1)$ *lies on it, if and only if* $\mathbf{p} \cdot \mathbf{u} = 0$.

Proof. The equation of the two circles are the following:

$$(x - v_1)^2 + (y - v_2)^2 = v_1^2 + v_2^2 - 1$$
$$x^2 + y^2 = 1$$

Expanding the first equation, we can simplify it:

$$x^2 - 2xv_1 + \cancel{v_1^2} + y^2 - 2yv_2 + \cancel{v_2^2} = \cancel{v_1^2} + \cancel{v_2^2} - 1$$
$$x^2 + y^2 = 1$$

Finally, we get the equation of the radical line of the circles: $-2xv_1 - 2yv_2 = 2 \Rightarrow -xv_1 - yv_2 = 1$. But the radical line is our original u line with the equation $xu_1 + yu_2 + 1 = 0$, therefore $v_1 = -u_1$ and $v_2 = -u_2$. ■

Remark 2.28 *According to (C-K∠), the angle of two lines* $\mathbf{u}(u_1, u_2, 1)$ *and* $\mathbf{v}(v_1, v_2, 1)$ *can be computed by their poles* $(-u_1, -u_2, 1)$ *and* $(-v_1, -v_2, 1)$. *With the defined bilinear form:*

$$\frac{-1 + (-u_1)(-v_1) + (-u_2)(-v_2)}{\sqrt{(-1 + (-u_1)^2 + (-u_2)^2)(-1 + (-v_1)^2 + (-v_2)^2)}} = \frac{-1 + u_1v_1 + u_2v_2}{\sqrt{(-1 + u_1^2 + u_2^2)(-1 + v_1^2 + v_2^2)}} \Rightarrow$$

$$\cos(\alpha) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\sqrt{\langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle}}.$$

2.5 Models of the Euclidean plane

– Horosphere: We consider the rays, orthogonal to the base plane in the Poincaré half-space model. This is a parallel line pencil. The corresponding cycles are horospheres, represented as planes, parallel to the base plane. Because this is a conformal model, the interior angel sum of any triangle on this plane is π. The geometry on this horosphere is Euclidean.

Theorem 2.29 *Any two horosphere are congruent to each other.*

– Hypercycle: Let O be a point on the Euclidean plane and K be a center of a unit ball, tangent to the plane at O. We consider a unit disk, such that its plane is parallel to the base plane, through K, as the Cayley-Klein model of the hyperbolic plane. The points of the model are the points of the unit disk, and the lines are the lines, through K and the corresponding hypercycles, represented as ellipses. We project this elliptic arc orthogonally onto the sphere (we get a great arc). Then we project it back to the base plane through K.

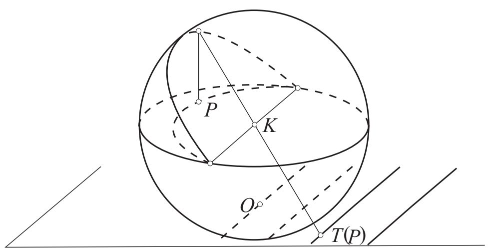


Figure 2.17: Hypercycle model

With the $d(A, B) := |\overline{T(A)T(B)}|$ metric, the congruence axioms will be true, and the Euclidean axiom of parallelism is obviously ture.

2.6 Area on the hyperbolic plane

Definition 2.30 *A triangle is called asymptotic/doubly asymptotic/triply asymptotic, if one/two/three vertex/vertices is/are boundary point(s).*

Theorem 2.31 *All the triply asymptotic triangles are congruent to each other.*

Definition 2.32 *Area is an isometry-invariante, non-negative, additive set function for simply polygons and $T(=\pi)$ is assigned to the triply asymptotic triangle.*

Theorem 2.33 *Any asymptotic triangle can be cur off into a pentagon.*

Proof. (Poincaré disk-model): Let $ABC\triangle$ be such that C is the ideal point and A is the center of the disk. Let D be an ideal point such that (ABD), and M be the footpoint of the perpendicular line to CD through A. Let A_1 be the reflection of B in the line AM and the intersection of BC and A_1D be M_1. Let the footpoints of the perpendicular line to DC through B and A_1 be Q and P respectively. If the reflection of BC in A_1P is DA_2, then the triangle $M_2A_1A_2\triangle/4/$ is congruent to $A_1M_1M_2\triangle/3/$ and $M_1M_2B\triangle$. Continuing this process, we get $ABQPA_1$ pentagon. ■

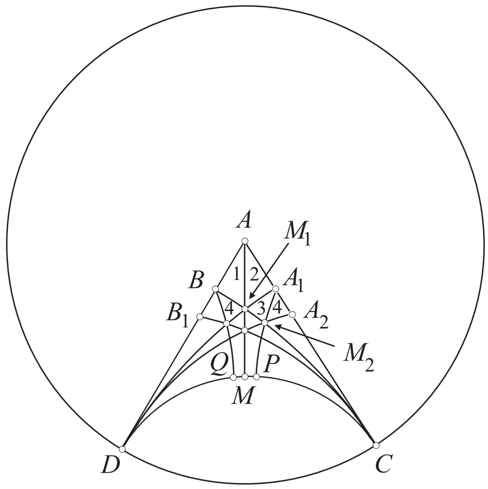


Figure 2.18: Lindberg method

Theorem 2.34 *If, the vertex angle of a doubly asymptotic triangle is α, then the area of it is $c(\pi-\alpha)$ $/c\in\mathbb{R}^+/$.*

Proof. Let $f(\phi)$ be the area, if $\phi=\pi-\alpha$ is the supplementary angle. The union of the two corresponding doubly asymptotic triangle is a triply asymptotic triangle (see Figure 2.19), therefore: $f(\phi)+f(\pi-\phi)=T$.

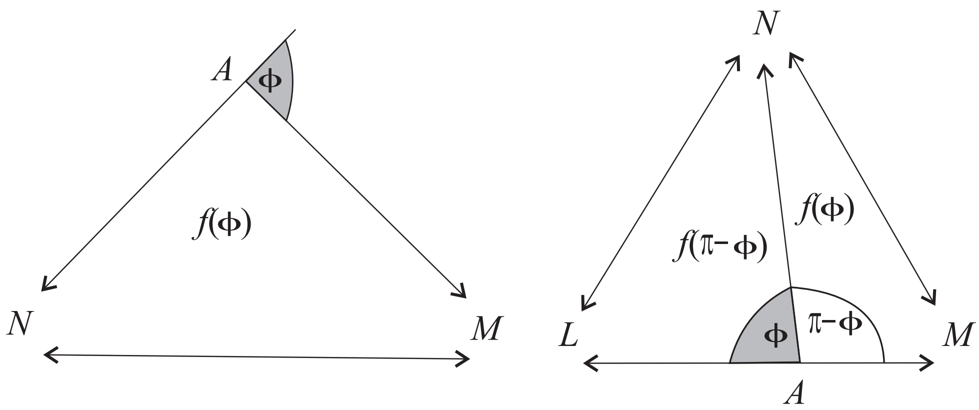


Figure 2.19: $f(\phi)$ is additive

Figure 2.20 shows us, that $T = f(\phi) + f(\varphi) + f(\pi - \phi - \varphi) \Rightarrow f(\phi) + f(\varphi) = f(\phi + \varphi)$

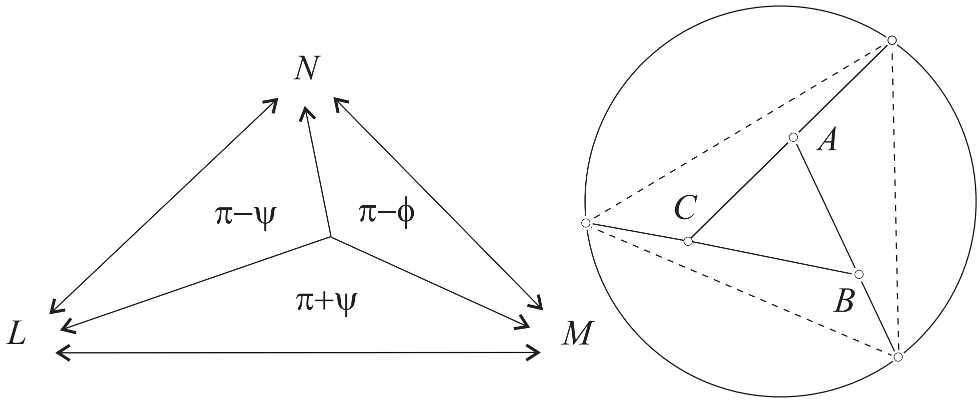


Figure 2.20: Area and defect

Our only solution is $f(x) = \lambda x$, since f is monotonously increasing and $f(1) = \lambda \Rightarrow f(n) = n\lambda$. Now, if $\frac{k}{n} \leq x \leq \frac{k+1}{n} \Rightarrow k \leq nx \leq k+1 \Rightarrow k\lambda \leq nf(x) \leq (k+1)\lambda \Rightarrow \frac{k}{n} \leq \frac{f(x)}{\lambda} \leq \frac{k+1}{n} \Rightarrow \left|x - \frac{f(x)}{\lambda}\right| \leq \frac{1}{n} \forall n \Rightarrow f(x) = \lambda x$. ■

Remark 2.35 *Because the triply asymptotic triangle is* $T \Rightarrow \lambda = \frac{T}{\pi}$*. Therefore, we choose the value of* T *be* π*.*

Theorem 2.36 *The area of any hyperbolic triangle is its defect.*

Proof. We make up the $ABC\triangle$ to a triply asymptotic triangle by expanding the sides cyclically (see Figure 2.20). The area of the three extra doubly asymptotic triangles are α, β and γ respectively. Therefore, the area can be expressed from the formula:

$$\pi = A(ABC) + \alpha + \beta + \gamma \Rightarrow A(ABC) = \pi - (\alpha + \beta + \gamma) = \delta. \quad ■$$

3 Classification of isometries

3.1 Isometries of the Euclidean plane

Definition 3.1 *Isometry is a distance-preserving bijective mapping of the space to itself.*

Claim 3.2 *The isometries of the space form a non-commutative group.*

Theorem 3.3 *Every isometry is a collineation.*

Proof. Let A, B and C be three collinear points, and their image be A', B' and C'. Assume, that $A'B'C'$ form a triangle. But $AC \cong A'C'$, $BC \cong B'C'$ and $AB \cong A'B' \Rightarrow |\overline{AC}| + |\overline{BC}| = |\overline{AB}| = |\overline{A'B'}| < |\overline{A'C'}| + |\overline{B'C'}|$. $\Rightarrow\!\!\Leftarrow$ ■

Theorem 3.4 *If, an isometry has two fix points, then the determined line is also fixed point-by-point (axis).*

Proof. Let $A = A'$ and $B = B'$ be true and P be a point on this line. Then P' must be also on the line and $AP \cong AP' \wedge BP \cong BP' \Rightarrow P = P'$. ■

Theorem 3.5 *If, an isometry has three fix points, then the determined plane is also fixed point-by-point (plane).*

Definition 3.6 *The reflection of the plane in a line t assign the P' point to P if $PP' \perp t$ and $PT \cong P'T$, where $T = PP' \cap t$.*

Theorem 3.7 *Reflection is an isometry.*

Proof. Let C and D be the footpoints of the perpendiculars to t from A and B respectively. Then $BCD\triangle \cong B'CD\triangle \Rightarrow BC \cong B'C \wedge BCD\angle B'CD\angle \Rightarrow ACB\angle \cong A'CB'\angle \Rightarrow ABC\triangle \cong A'B'C\triangle \Rightarrow AB \cong A'B'$ ■

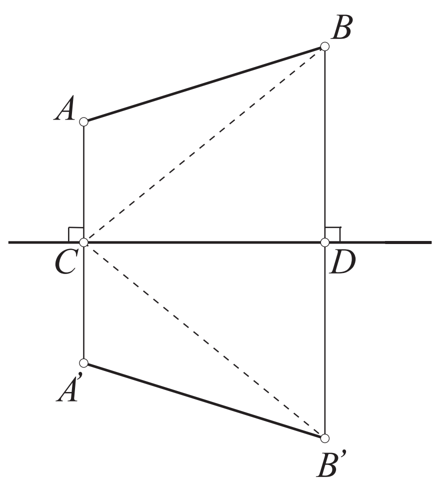


Figure 3.1: Hypercycle model

Claim 3.8 *The composition of a reflection to itself is the identical mapping.*

Theorem 3.9 *If, a non-identical isometry has exactly two fix points, then it is a reflection in the determined line.*

Proof. Let A and B be the fix points and P be an arbitrary point. Then $AP \cong AP' \wedge BP \cong BP' \Rightarrow P = P'$, or AB is the perpendicular bisector of $PP' \Rightarrow d(P, AB) = d(P', AB)$ and $PP' \perp AB$. ■

Theorem 3.10 *If, a non-identical R isometry has exactly one fix point, then it is the composition of two reflections, through this fix point.*

Proof. Let P be an arbitrary point and A be the fix point. Then $AP \cong AP'$, therefore A lies on the perpendicular bisector (t_1) of PP'. The composition of the original isometry and the reflection in this line has two fix points, therefore it is a reflection in a line t_2. Then $R \circ t_1 = t_2 \Rightarrow R = t_2 \circ t_1^{-1} = t_2 \circ t_1$. ■

Theorem 3.11 *If, a non-identical isometry T has no fix point, then it is the composition of either two or three reflections.*

Proof. Let P be an arbitrary point and P' be its image. Then $T \circ t_1$ has either one, or two fix points, where t_1 is the reflection to the perpendicular bisector of PP'.

1. $T \circ t_1 = R = t_2 \circ t_3 \Rightarrow T = t_2 \circ t_3 \circ t_1^{-1} = t_2 \circ t_3 \circ t_1$
2. $T \circ t_1 = t_2 \Rightarrow T = t_2 \circ t_1^{-1} = t_2 \circ t_1$ ■

Theorem 3.12 *Any isometry of the plane is the composition of at most three reflections in a line.*

Proof. It follows from the above theorems trivially, since any isometry has either 0, 1, or 2 fix points or all the points on the plane is fixed point-by-point. ■

Lemma 3.13 *The composition of two reflections can be considered as either a rotation (intersecting axis) or a translation (parallel axis).*

Proof.
– Intersecting axis: Let A be the intersection of the two lines t_1 and t_2. Then $PAt_1\angle \cong P'At_1\angle$ and $P'At_2\angle \cong P''At_2\angle$, therefore $PAP''\angle = 2t_1t_2\angle = 2\alpha \Rightarrow$ Rotation around A by 2α.
– Parallel axis: Now, we can say that $d(P, t_1) = d(P', t_1)$ and $d(P', t_2) = d(P'', t_2)$ and P, P', P'' are collinear points, therefore $d(P, P'') = d(t_1, t_2) = 2d \Rightarrow$ Translation, perpendicular to t_i with $2d$. ■

Remark 3.14 *In the case of the rotation, only the angle of the lines matters, in the case of the translation, only the distance and the direction matters.*

Theorem 3.15 *Every isometry of the Euclidean plane belongs to one of the following 5 groups:*

1. *Identity (0)*
2. *Reflection (1)*
3. *Rotation (2a)*
4. *Translation (2b)*
5. *Glide reflection (3)*

Proof. We have a full discussion for 2 reflections.

1. case $t_1 \cap t_2 \neq \emptyset$

   (a) $t_1 \cap t_2 = t_2 \cap t_3$

   Then we can consider the $t_2 \circ t_3$ as a rotation around this point, and we can rotate the lines around this center to get $t_1 = t_2'$. Then $t_1 \circ t_2 \circ t_3 = t_1 \circ t_2' \circ t_3' = t_1 \circ t_1^{-1} \circ t_3' = t_3'$.

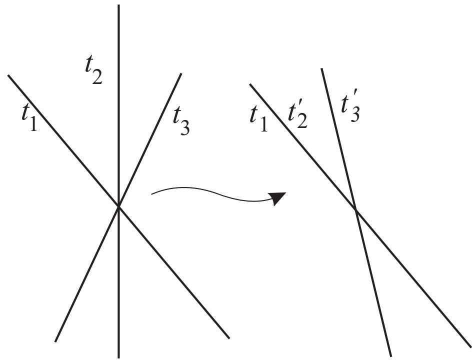


Figure 3.2: Case 1. (a)

   (b) $t_1 \cap t_2 \neq t_2 \cap t_3$

   First, we rotate t_1 and t_2 around their intersection O until $t_2' \perp t_3$. Then we rotate t_2' and t_3 around their intersection K until $t_2'' \parallel t_1'$. Then $t_1' \circ t_2''$ is a translation and $t_1' \circ t_2'' \circ t_3'$ is a glide reflection.

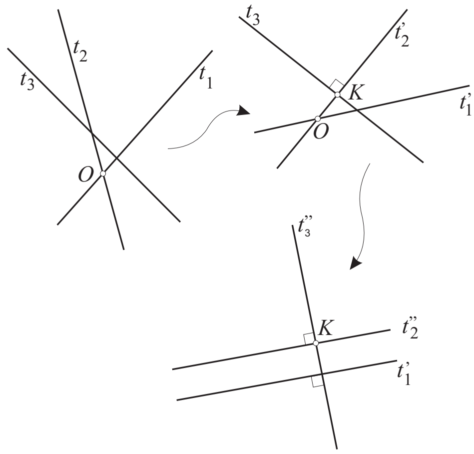


Figure 3.3: Case 1. (b)

2. case $t_1 \cap t_2 = \emptyset \Rightarrow t_1 \parallel t_2$

   (a) $t_1 \parallel t_2 \parallel t_3$

   Then we can translate t_2 and t_3 along their perpendicular line until $t_1 = t_2'$ then their composition will be the identity, therefore only t_3 remains, so this is a translation.

   (b) $t_1 \parallel t_2 \nparallel t_3$

   First, we rotate t_2 and t_3 around their intersection until $t_2' \perp t_1$. Then we rotate t_1 and t_2' around their intersection until $t_1' \parallel t_3'$. Finally, we rotate t_2'' and t_3' around their intersection, until $t_1' \parallel t_2'''$. Then $t_1' \circ t_2''' \circ t_3''$ is a glide reflection. ■

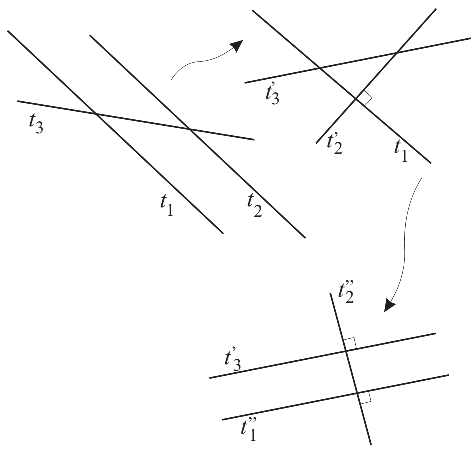


Figure 3.4: Case 2. (b)

Remark 3.16 *Two reflections are commutative to each other if their axis are orthogonal to each other.*

3.2 Isometries of the Euclidean space

Theorem 3.17 *Any isometry of the space is the composition of at most four reflections in a plane.*

Theorem 3.18 *Every isometry of the Euclidean space belongs to one of the following 7 groups:*

1. *Identity (0)*
2. *Reflection (1)*
3. *Rotation (2a)*
4. *Translation (2b)*
5. *Glide reflection (3a)*
6. *Improper rotation (3b)*
7. *Screw displacement (4)*

where

5. *Glide reflection: composition of a translation and a reflection*
6. *Improper rotation: composition of a rotation and a reflection*
7. *Screw displacement: composition of a rotation and a translation*

Definition 3.19 *The reflection in a line in the space means a rotation around this line by* $180°$. *This can be represented as a reflection in two orthogonal planes through the line.*

Lemma 3.20 *The composition of two reflections in a line is either rotation or translation or screw displacement, if the two lines are intersecting, parallel or skew lines.*

Proof.
Assume that $e = t_1 \circ t_2$, $f = t_3 \circ t_4$.
1. case: The lines e and f determine a plan
We choose the positions of t_i such that, t_2 and t_3 coincide the determined plane and t_1 and t_4 are orthogonal to it. Then $t_2 \circ t_3$ is identity and if $e \parallel f$, then $t_1 \parallel t_4$, otherwise $t_1 \nparallel t_4$.
2. case: The lines e and f are skew lines
Let n be the perpendicular transverse of e and f. $t_1 := (e, n) \Rightarrow t_2 \perp n$, $t_3 := (f, n) \Rightarrow t_4 \perp n$. Since $t_2 \parallel t_4 \wedge t_3 \perp t_4 \Rightarrow t_3 \perp t_2 \Rightarrow t_1 \circ t_2 \circ t_3 \circ t_4 = (t_1 \circ t_3) \circ (t_2 \circ t_4)$. But n lies both on t_1 and t_3 therefore they determine a rotation, and $t_2 \parallel t_4$ then they determine a translation. The composition of these isometries is a skew displacement. ■

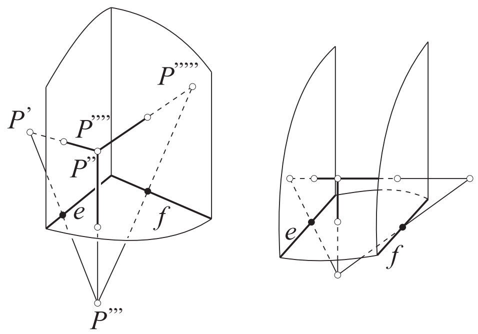


Figure 3.5: 1. case

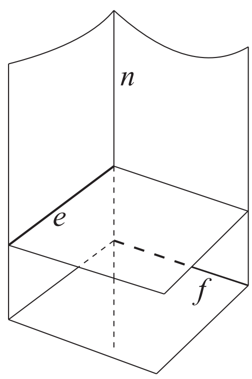


Figure 3.6: 2. case

Lemma 3.21 *The composition of two translations is also translation.*

Proof. Let $t_1 \parallel t_2$ and $t_3 \parallel t_4$ be two translation. If, $t_2 \parallel t_3$, then it is trivial. Otherwise $t_2 \cap t_3 \neq \emptyset \Rightarrow t_2 \cap t_3 = e$. If, we rotate t_2 and t_3 around e by 90°, then $t_2' \perp t_1$ and $t_3' \perp t_4$. Then $= t_1, t_2'$ and t_3', t_4 determines two reflections in a line with parallel axis, therefore it is a translation. ■

Lemma 3.22 *The composition of two rotations with orthogonal axis is a screw displacement.*

Proof. Let t' and t'' be the axis of the rotations and n be their perpendicular transverse line. Then $t' \perp (t'', n)$ and $t'' \perp (t', n)$. Let t_1 and t_2 be reflections such that $t_1 \cap t_2 = t'$ and $t_2 = (t', n)$, furthermore t_3 and t_4 be reflections such that $t_3 \cap t_4 = t''$ and $t_3 = (t'', n)$. Since $t_2 \perp t_3 \Rightarrow t_1 \circ t_2 \circ t_3 \circ t_4 = (t_1 \circ t_3) \circ (t_2 \circ t_4)$. Let e and f be the intersections of t_1, t_3 and t_2, t_4 respectively. But $t_1 \perp t_3$ and $t_2 \perp t_4$, therefore this isometry is can be represented as the composition of two reflections in a line, which axis are in skew position $\Rightarrow$ screw displacement. ■

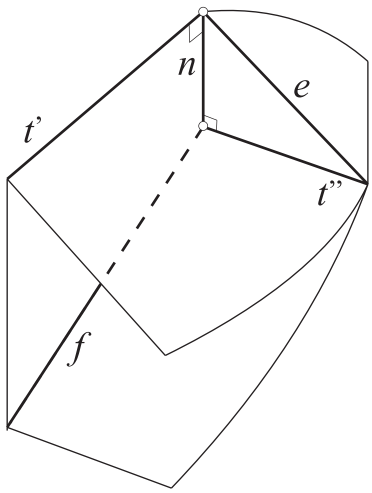


Figure 3.7: Proof of Lemma 3.22

Proof of Theorem 3.18 We have a full discussion for 2 reflections. Composition of at least three reflections:

1. case: The three plane have a common perpendicular plane:
   The orbit of any point lies on the plane, parallel to this orthogonal plane $\Rightarrow$ using the planar case, it is the composition of 3 reflections in line $\Rightarrow$ either reflection or glide reflection $\Rightarrow$ the spatial transformation is also either reflection or glide reflection.

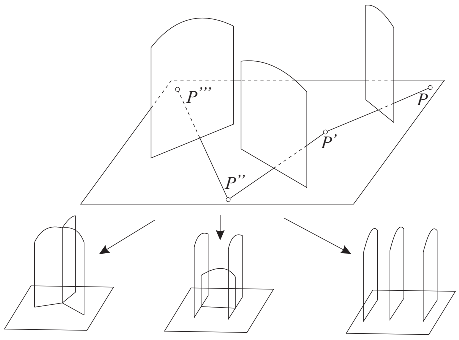


Figure 3.8: 1. case

2. case: If, plane 1 and plane 2 intersect each other in the line m and plane 3 does not intersect m, then the plane orthogonal to m will be orthogonal to plane 3 as well $\Rightarrow$ 1. case. We may assume that plane 3 intersects m in the point M. First we rotate plane 1 and 2 around m such that $2' \perp 3$. Then we rotate $2'$ and $3'$ around their intersection such that $1' \perp 3'$. Finally we get an improper rotation.

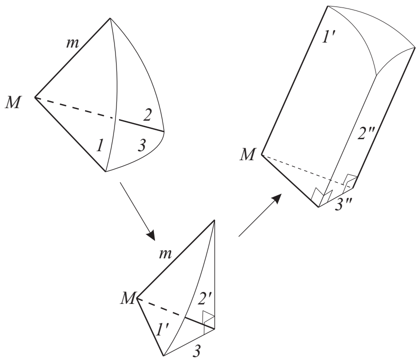


Figure 3.9: 2. case

3. case: If, plane 1 and 2 are parallel to each other, then plane 3 is either parallel them both ⇒ reflection, or plane 3 intersects them in parallel lines. Then the plane, orthogonal to the lines will be orthogonal to all the planes ⇒ 1. case

4. We can assume, that the composition of the first 3 reflections is either glide reflection or improper rotation.
   If, plane 4 is parallel to the third plane of an improper rotation, then it is a screw displacement by definition.
   If, m is the intersection of plane 3 and 4, then this isometry is the composition of two rotation and by lemma, it is a screw displacement.
   Now, we can assume, that the composition of the first 3 reflections is a glide reflection.
   If, plane 4 is parallel to plane 3, then this is the composition of two translations and by lemma, this is another translation.
   If, plane 4 intersects plane 3, then we rotate plane 2 and 3 around their intersection line such that

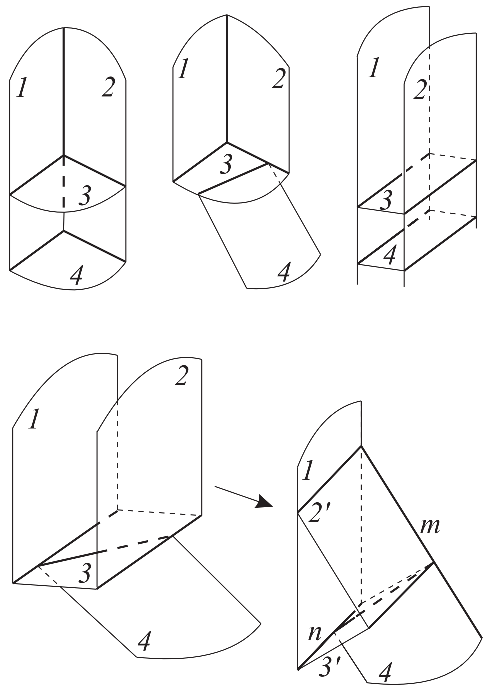


Figure 3.10: 4. case

   $3' \perp 4$. Then plane 3’ is orthogonal to both plane 2’ and plane 4 ⇒ plane 3’ is orthogonal to their intersection line m. Let n be the intersection line of plane 1 and plane 3’. Then $n \perp m$ and $1 \circ 2 \circ 3 \circ 4 = 1 \circ 2' \circ 3' \circ 4$, but $2' \perp 3' \Rightarrow 1 \circ 2' \circ 3' \circ 4 = (1 \circ 3') \circ (2' \circ 4)$. Then this isometry is the composition of two rotation around orthogonal axis, therefore this is a screw displacement. ■

4 Analytic geometry

4.1 Vectors in the Euclidean space

Definition 4.1 *The representing point pairs of a translation called vector.*

Definition 4.2 *The sum of two vectors is a vector, that we get by the parallelogram rule.*

Definition 4.3 *Let $t > 0$, $t \in \mathbb{R}$, then the multiplication of $\overrightarrow{PP'}$ by t is $\overrightarrow{PP''}$ if $\left|\overrightarrow{PP''}\right| = t\left|\overrightarrow{PP'}\right|$ and $(P'PP'')$ is not true. If $t = 0$, then $\overrightarrow{PP''} = \underline{0}$. If $t < 0$, then $\overrightarrow{PP''}$ is such that $\left|\overrightarrow{PP''}\right| = -t\left|\overrightarrow{PP'}\right|$ and $(P'PP'')$ is ture.*

Lemma 4.4 *The vectors of the space with the defined addition and scalar multiplication form a vector space.*

Definition 4.5 *Let $\underline{a}$ and $\underline{b}$ be vectors in the space, then $\langle \underline{a}, \underline{b} \rangle = |\underline{a}||\underline{b}|\cos(\gamma)$ called the dot product.*

Remark 4.6 $\langle \underline{a}, \underline{b} \rangle = 0 \Leftrightarrow a \perp b$, $\langle \underline{a}, \underline{b} \rangle > 0 \Leftrightarrow \gamma < \frac{\pi}{2}$, $\langle \underline{a}, \underline{b} \rangle < 0 \Leftrightarrow \gamma > \frac{\pi}{2}$.

Theorem 4.7 *Dot product is symmetric, positive definite, bilinear function.*

Lemma 4.8 *Let $\underline{a}$ and $\underline{v}$ be vectors. Then $\underline{v} = \underline{v}_\perp + \underline{v}_\parallel$, where $\underline{a} \parallel \underline{v}_\parallel$, $\underline{a} \perp \underline{v}_\perp$ and $\underline{v}_\parallel = \dfrac{\langle \underline{a}, \underline{v} \rangle}{\langle \underline{a}, \underline{a} \rangle}\underline{a}$, $\underline{v}_\perp = \underline{v} - \underline{v}_\parallel$.*

Proof. Let α be the angle between $\underline{a}$ and $\underline{v}$. Then $\dfrac{|\underline{v}_\parallel|}{|\underline{v}|} = \cos\alpha \Rightarrow |\underline{v}_\parallel| = |\underline{v}|\cos\alpha$, but $\langle \underline{a}, \underline{v} \rangle = |\underline{a}||\underline{v}|\cos\alpha$, therefore $|\underline{v}_\parallel| = \dfrac{\langle \underline{a}, \underline{v} \rangle}{|\underline{a}|}$. Now, we need a unit vector in direction of $\underline{a}$ to obtain $\underline{v}_\parallel$: $\underline{a}_u = \dfrac{\underline{a}}{|\underline{a}|}$. So that $\underline{v}_\parallel = |\underline{v}_\parallel|\underline{a}_u = \dfrac{\langle \underline{a}, \underline{v} \rangle}{|\underline{a}|^2}\underline{a} = \dfrac{\langle \underline{a}, \underline{v} \rangle}{\langle \underline{a}, \underline{a} \rangle}\underline{a}$. ■

Definition 4.9 *Let $\underline{a}$ and $\underline{b}$ be vectors, then $\underline{a} \times \underline{b}$ is the cross product of $\underline{a}$ and $\underline{b}$, where $|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin\gamma$, $\underline{a} \times \underline{b}$ is orthogonal both $\underline{a}$ and $\underline{b}$ and the direction of $\underline{a} \times \underline{b}$ is given by the right-hand rule.*

Theorem 4.10 *Cross product is antisymmetric bilinear function.*

Remark 4.11 $\underline{a} \times \underline{b} = \underline{0} \Leftrightarrow \underline{a} \parallel \underline{b}$*, and* $|\underline{a} \times \underline{b}|$ *is equal to the area of the spanned parallelogram.*

Definition 4.12 *Let* $\underline{a}$*,* $\underline{b}$ *and* $\underline{c}$ *be vectors in the space. Then the triple product of them is* $\underline{a} \cdot \underline{b} \cdot \underline{c} = \langle \underline{a} \times \underline{b}, \underline{c} \rangle$ *.*

Theorem 4.13 *Triple product is invariant under a circular shift:* $\underline{a} \cdot \underline{b} \cdot \underline{c} = \langle \underline{a} \times \underline{b}, \underline{c} \rangle = \langle \underline{b} \times \underline{c}, \underline{a} \rangle = \langle \underline{c} \times \underline{a}, \underline{b} \rangle$ *.*

Proof. This is true because of the geometric meaning. $\underline{a} \cdot \underline{b} \cdot \underline{c}$ is the (signed) volume of the spanned parallelepiped. The area, spanned by $\underline{a}$ and $\underline{b}$ is exactly $|\underline{a} \times \underline{b}|$ and $\frac{h}{|\underline{c}|} = \cos(\underline{a} \times \underline{b}, \underline{c}) \Rightarrow h = |\underline{c}| \cos\gamma$. Then the volume of the spanned parallelepiped is $V = |\underline{a} \times \underline{b}||\underline{c}| \cos\gamma = \langle \underline{a} \times \underline{b}, \underline{c} \rangle = \underline{a} \cdot \underline{b} \cdot \underline{c}$. ■

Theorem 4.14 (Lagrange's formula) $(\underline{a} \times \underline{b}) \times \underline{c} = \langle \underline{a}, \underline{c} \rangle \underline{b} - \langle \underline{b}, \underline{c} \rangle \underline{a}$

Proof. $(\underline{a} \times \underline{b}) \times \underline{c} \perp \underline{a} \times \underline{b} \Rightarrow (\underline{a} \times \underline{b}) \times \underline{c}$ is in the plane of $\underline{a}$ and $\underline{b} \Rightarrow (\underline{a} \times \underline{b}) \times \underline{c} = \alpha \underline{a} + \beta \underline{b}$
Now, we consider the dot product of both sides with $\underline{b} \times \underline{c}$:
Left side: $\langle (\underline{a} \times \underline{b}) \times \underline{c}, \underline{b} \times \underline{c} \rangle = \langle \underline{c} \times (\underline{b} \times \underline{c}), \underline{a} \times \underline{b} \rangle$. Now, if $\underline{v} := \underline{c} \times (\underline{b} \times \underline{c})$, then $|\underline{v}| = |\underline{c}||\underline{b} \times \underline{c}|$, since $\underline{c} \perp \underline{b} \times \underline{c}$. But than $|\underline{v}| = |\underline{c}|^2 |\underline{b}| \sin\gamma$, where γ is the angle of $\underline{b}$ and $\underline{c}$. If we decompose $\underline{b}$ into parallel and orthogonal elements respected to $\underline{c}$, then $|\underline{b}_\perp| = |\underline{b}| \sin\gamma$, therefore $\frac{|\underline{v}|}{|\underline{c}|^2} = |\underline{b} \perp|$. It can be seen, that $\underline{v} \parallel \underline{b}_\perp \Rightarrow \underline{v} = |\underline{c}|^2 \underline{b}_\perp = |\underline{c}|^2 (\underline{b} - \underline{b}_\parallel) = |\underline{c}|^2 \left(\underline{b} - \frac{\langle \underline{b}, \underline{c} \rangle}{|\underline{c}|^2} \underline{c} \right) = |\underline{c}|^2 \underline{b} - \langle \underline{b}, \underline{c} \rangle \underline{c}$. Finally, $\langle \underline{c} \times (\underline{b} \times \underline{c}), \underline{a} \times \underline{b} \rangle = \langle |\underline{c}|^2 \underline{b} - \langle \underline{b}, \underline{c} \rangle \underline{c}, \underline{a} \times \underline{b} \rangle = 0 - \langle \underline{b}, \underline{c} \rangle \langle \underline{c}, \underline{a} \times \underline{b} \rangle = - \langle \underline{b}, \underline{c} \rangle (\underline{a} \cdot \underline{b} \cdot \underline{c})$.
Right side: $\alpha \langle \underline{a}, \underline{b} \times \underline{c} \rangle = \alpha (\underline{a} \cdot \underline{b} \cdot \underline{c})$ and $\underline{b} \perp \underline{b} \times \underline{c} \Rightarrow \beta \langle \underline{b}, \underline{b} \times \underline{c} \rangle = 0$
Finally, we get that $\alpha = - \langle \underline{b}, \underline{c} \rangle$. Similarly, $\beta = \langle \underline{a}, \underline{c} \rangle$. ■

Theorem 4.15 (Jacobi's formula) $(\underline{a} \times \underline{b}) \times \underline{c} + (\underline{b} \times \underline{c}) \times \underline{a} + (\underline{c} \times \underline{a}) \times \underline{b} = 0$

Proof. $(\underline{b} \times \underline{c}) \times \underline{a} = -\underline{a} \times (\underline{b} \times \underline{c})$ and $(\underline{c} \times \underline{a}) \times \underline{b} = -(\underline{a} \times \underline{c}) \times \underline{b}$ due to the antisymmetry.
$(\underline{a} \times \underline{b}) \times \underline{c} - \underline{a} \times (\underline{b} \times \underline{c}) = \langle \underline{a}, \underline{c} \rangle \underline{b} - \langle \underline{b}, \underline{c} \rangle \underline{a} + \langle \underline{a}, \underline{b} \rangle \underline{c} - \langle \underline{a}, \underline{c} \rangle \underline{b} = \langle \underline{a}, \underline{b} \rangle \underline{c} - \langle \underline{b}, \underline{c} \rangle \underline{a} = (\underline{a} \times \underline{c}) \times \underline{b}$
■

Lemma 4.16 $\langle \underline{a} \times \underline{b}, \underline{a} \times \underline{b} \rangle = \langle \underline{a}, \underline{a} \rangle \langle \underline{b}, \underline{b} \rangle - \langle \underline{a}, \underline{b} \rangle^2$

Proof. $\langle \underline{a} \times \underline{b}, \underline{a} \times \underline{b} \rangle = |\underline{a} \times \underline{b}|^2 = |\underline{a}|^2 |\underline{b}|^2 \sin^2\gamma = |\underline{a}|^2 |\underline{b}|^2 (1 - \cos^2\gamma) = |\underline{a}|^2 |\underline{b}|^2 - |\underline{a}|^2 |\underline{b}|^2 \cos^2\gamma = \langle \underline{a}, \underline{a} \rangle \langle \underline{b}, \underline{b} \rangle - \langle \underline{a}, \underline{b} \rangle^2$ ■

Theorem 4.17

$$\langle \underline{a}, \underline{b} \rangle = \underline{a}^T \underline{b}, \qquad \underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \qquad \underline{a} \cdot \underline{b} \cdot \underline{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Point: (x, y, z)

Line: Let P_0 be a point on the line l and $\underline{v}$ its direction. Then $P \in l \Leftrightarrow \exists t \in \mathbb{R} : P = P_0 + t\underline{v}$ $\Rightarrow x = x_0 + tv_1,\ y = y_0 + tv_2,\ z = z_0 + tv_3$ or $\dfrac{x - x_0}{v_1} = \dfrac{y - y_0}{v_2} = \dfrac{z - z_0}{v_3}$ if $v_i \neq 0$.

Plane: Let P_0 be a point on the plane α and $\underline{n}(A, B, C)$ be orthogonal to the plane. Then $P \in \alpha \Leftrightarrow \overrightarrow{P_0P} \perp \underline{n} \Leftrightarrow \left\langle \overrightarrow{P_0P}, \underline{n} \right\rangle = 0 \Leftrightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = Ax + By + Cz + D = 0$, where $D = -(Ax_0 + By_0 + Cz_0)$.

4.2 Transformation of $\mathbf{E}^3$ with one fix point (origin)

1. Reflection in a plane through the origin

   $\underline{p}' = \underline{p} + \overrightarrow{PP'} = \underline{p} + 2\overrightarrow{PT} = \underline{p} - 2\underline{p}_{\|} = \underline{p} - 2\dfrac{\left\langle \underline{p}, \underline{n} \right\rangle}{\langle \underline{n}, \underline{n} \rangle}\underline{n}$,

   where $\underline{n} = (A, B, C)^T$ such that $|\underline{n}| = 1$.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - 2\begin{bmatrix} Ax & By & Cz \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} =$$

$$\begin{bmatrix} 1 - 2A^2 & -2AB & -2AC \\ -2AB & 1 - 2B^2 & -2BC \\ -2AC & -2BC & 1 - 2C^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_\alpha \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

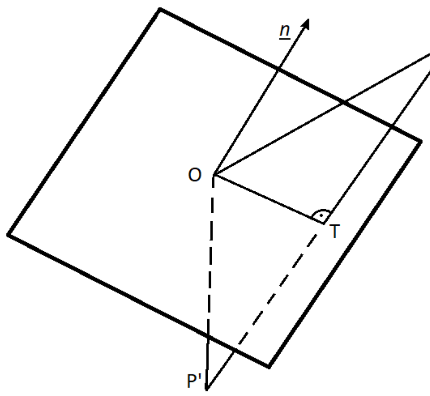


Figure 4.1: Reflection in plane

2. Projection to a plane through the origin

   $\underline{p}' = \underline{p} - \underline{p}_{\|} \Rightarrow$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 - A^2 & -AB & -AC \\ -AB & 1 - B^2 & -BC \\ -AC & -BC & 1 - C^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = P_\alpha \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

   where $\underline{n} = (A, B, C)^T$ such that $|\underline{n}| = 1$.

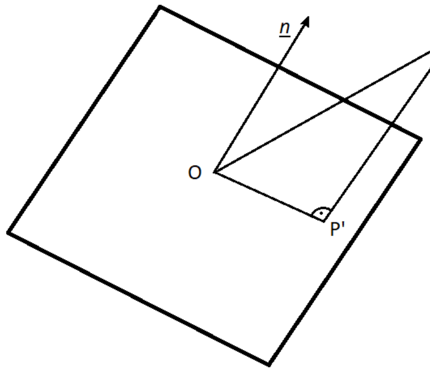


Figure 4.2: Projection in plane

3. Reflection in a line through the origin

   First, we reflect P in the plane, orthogonal to the line l to obtain P^*, then we reflect P^* in the origin to obtain P'. Let $\underline{v}_l = \underline{n} = (A, B, C)^T$, such that $|\underline{v}_e| = 1$.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = -\underline{\underline{I}} \cdot \begin{bmatrix} 1-2A^2 & -2AB & -2AC \\ -2AB & 1-2B^2 & -2BC \\ -2AC & -2BC & 1-2C^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$= \begin{bmatrix} 2A^2-1 & 2AB & 2AC \\ 2AB & 2B^2-1 & 2BC \\ 2AC & 2BC & 2C^2-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_l \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

4. Projection to a line through the origin
   $\overrightarrow{OP} = \underline{p} = \overrightarrow{OP'} + \overrightarrow{OP^*} \Rightarrow \underline{p}' = \underline{p} - \overrightarrow{OP^*}$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underline{\underline{I}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1-A^2 & -AB & -AC \\ -AB & 1-B^2 & -BC \\ -AC & -BC & 1-C^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$= \begin{bmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = P_e \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{v}_e \underline{v}_e^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

5. Left cross product with a fix vector

$$\underline{p}' = \underline{n} \times \underline{p} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A & B & C \\ x & y & z \end{vmatrix} = \begin{bmatrix} Bz - Cy \\ Cx - Az \\ Ay - Bx \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -C & B \\ C & 0 & -A \\ -B & A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = C_{\underline{n}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

6. Rotation around a line, through the origin
   $\underline{p}' = P_l(\underline{p}) + \cos\phi P_\alpha(\underline{p}) + \sin\phi C_{\underline{n}}(\underline{p}) \Rightarrow$

$$\begin{bmatrix} A^2 & AB & AC \\ AB & B^2 & BC \\ AC & BC & C^2 \end{bmatrix} + \sin\phi \begin{bmatrix} 0 & -C & B \\ C & 0 & -A \\ -B & A & 0 \end{bmatrix} +$$

$$+ \cos\phi \begin{bmatrix} 1-A^2 & -AB & -AC \\ -AB & 1-B^2 & -BC \\ -AC & -BC & 1-C^2 \end{bmatrix} = R_{\underline{n}}^{\phi}$$

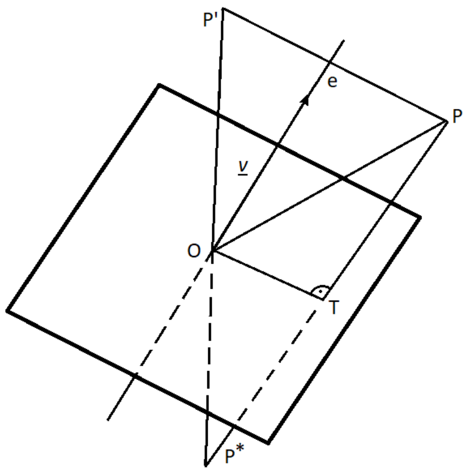


Figure 4.3: Reflection in line

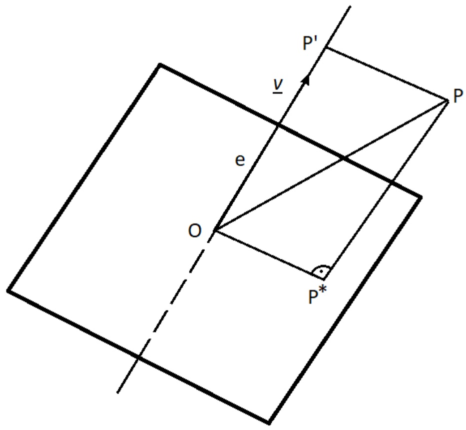


Figure 4.4: Projection in line

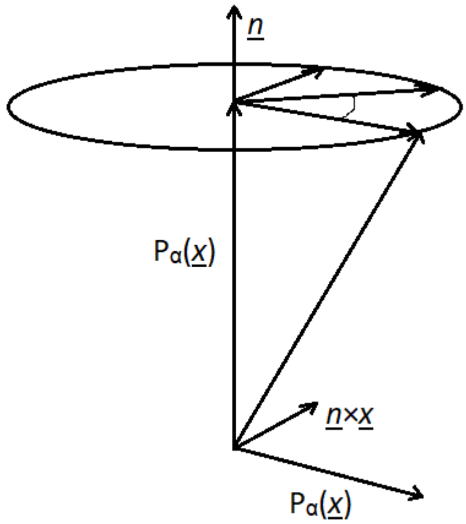


Figure 4.5: Rotation around a line

Definition 4.18 *The homogeneous coordinates of a spatial point P is the equivalence class of the assigned point quartet.*

$$P(\underline{p}) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \hookrightarrow \underline{\widetilde{p}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \sim \left\{ \alpha \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \setminus \{0\} \right\}.$$

Q is an ideal point, if $Q \sim [x, y, z, 0]^T$, but $\nexists R \sim [0, 0, 0, 0]^T$.

7. Translation with $\underline{t}$

$$\underline{p}' = \underline{p} + \underline{t} = \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = T_{\underline{t}} \sim \left\{ \alpha \cdot \begin{bmatrix} \underline{\underline{I}} & \underline{t} \\ \underline{0}^T & 1 \end{bmatrix}, \alpha \in \mathbb{R} \setminus \{0\} \right\}$$

8. Linear transformations

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} \Rightarrow \begin{bmatrix} \underline{x}' \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{\underline{A}} & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix}$$

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{b} \Rightarrow \begin{bmatrix} \underline{x}' \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{\underline{A}} & \underline{b} \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix}$$

9. Point on a line

   Let $X(\underline{x})$ and $Y(\underline{y})$ be two points on a line l. Then $U(\underline{u}) \in l \Leftrightarrow \exists \alpha \underline{u} = \underline{y} + \alpha(\underline{x} - \underline{y}) = \alpha \underline{x} + (1 - \alpha)\underline{y})$. Now let α and β be real numbers, then:

$$\alpha \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix} + \beta \begin{bmatrix} \underline{y} \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \underline{x} + \beta \underline{y} \\ \alpha + \beta \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\alpha+\beta}\underline{x} + \frac{\beta}{\alpha+\beta}\underline{y} \\ 1 \end{bmatrix}$$

10. Rotation around a line

   (a) Translation of a point of the line to the origin: $\begin{bmatrix} \underline{\underline{I}} & -\underline{t} \\ \underline{0}^T & 1 \end{bmatrix}$

   (b) Rotation around a line through the origin: $\begin{bmatrix} \underline{\underline{R}}_{\underline{n}}^{\alpha} & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix}$

   (c) Inverse translation: $\begin{bmatrix} \underline{\underline{I}} & \underline{t} \\ \underline{0}^T & 1 \end{bmatrix}$

$$\begin{bmatrix} \underline{\underline{I}} & \underline{t} \\ \underline{0}^T & 1 \end{bmatrix} \cdot \begin{bmatrix} \underline{\underline{R}}_{\underline{n}}^{\alpha} & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix} \cdot \begin{bmatrix} \underline{\underline{I}} & -\underline{t} \\ \underline{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \underline{\underline{I}} & \underline{t} \\ \underline{0}^T & 1 \end{bmatrix} \cdot \begin{bmatrix} \underline{\underline{R}}_{\underline{n}}^{\alpha} & -\underline{\underline{R}}_{\underline{n}}^{\alpha}\underline{t} \\ \underline{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \underline{\underline{R}}_{\underline{n}}^{\alpha} & \underline{t} - \underline{\underline{R}}_{\underline{n}}^{\alpha}\underline{t} \\ \underline{0}^T & 1 \end{bmatrix}$$

Definition 4.19 *If, X, Y and Z are collinear points such that $\underset{\sim}{\underline{z}} = \alpha\underset{\sim}{\underline{x}} + \beta\underset{\sim}{\underline{y}}$, then $(X,Y,Z) := \dfrac{\beta}{\alpha}$ is called division ratio. If, X, Y, Z and W are collinear points such that $\underset{\sim}{\underline{z}} = \alpha_1\underset{\sim}{\underline{x}} + \beta_1\underset{\sim}{\underline{y}}$ and $\underset{\sim}{\underline{w}} = \alpha_2\underset{\sim}{\underline{x}} + \beta_2\underset{\sim}{\underline{y}}$ then $(X,Y,Z,W) := \dfrac{\beta_1}{\alpha_1} : \dfrac{\beta_2}{\alpha_2} = \dfrac{(X,Y,Z)}{(X,Y,W)}$ is called cross ratio.*

Lemma 4.20 *The linear transformation $\widetilde{A} = \begin{bmatrix} \underline{\underline{A}} & t \\ \underline{u}^T & 1 \end{bmatrix}$ preserves cross ration and preserves division ratio if and only if $\underline{u} = \underline{0}$.*

Proof. $\underset{\sim}{\underline{z}} = \alpha_1\underset{\sim}{\underline{x}} + \beta_1\underset{\sim}{\underline{y}} \Rightarrow \widetilde{A\underset{\sim}{\underline{z}}} = \alpha_1\widetilde{A}\underset{\sim}{\underline{x}} + \beta_1\widetilde{A}\underset{\sim}{\underline{y}} = \alpha_1 \begin{bmatrix} \underline{\underline{A}} & t \\ \underline{u}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix} + \beta_1 \begin{bmatrix} \underline{\underline{A}} & t \\ \underline{u}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{y} \\ 1 \end{bmatrix} =$

$$\alpha_1 \begin{bmatrix} \underline{\underline{A}}\,\underline{x} + t \\ \underline{u}^T\underline{x} + 1 \end{bmatrix} + \beta_1 \begin{bmatrix} \underline{\underline{A}}\,\underline{y} + t \\ \underline{u}^T\underline{y} + 1 \end{bmatrix} = \frac{\alpha_1}{\underline{u}^T\underline{x}+1}\widetilde{A(\underset{\sim}{\underline{x}})} + \frac{\beta_1}{\underline{u}^T\underline{y}+1}\widetilde{A(\underset{\sim}{\underline{y}})} \Rightarrow$$

$$(\widetilde{A(\underset{\sim}{\underline{x}})}, \widetilde{A(\underset{\sim}{\underline{y}})}, \widetilde{A(\underset{\sim}{\underline{z}})}) = \frac{\frac{\beta_1}{\underline{u}^T\underline{y}+1}}{\frac{\alpha_1}{\underline{u}^T\underline{x}+1}} = \frac{\beta_1}{\alpha_1} \Leftrightarrow \underline{u}^T\underline{x} = \underline{u}^T\underline{y} \Leftrightarrow \underline{u} = \underline{0} \text{ and}$$

$$(\widetilde{A(\underset{\sim}{\underline{x}})}, \widetilde{A(\underset{\sim}{\underline{y}})}, \widetilde{A(\underset{\sim}{\underline{z}})}, \widetilde{A(\underset{\sim}{\underline{w}})}) = \frac{\frac{\beta_1}{\underline{u}^T\underline{y}+1}}{\frac{\alpha_1}{\underline{u}^T\underline{x}+1}} : \frac{\frac{\beta_2}{\underline{u}^T\underline{y}+1}}{\frac{\alpha_2}{\underline{u}^T\underline{x}+1}} = \frac{\beta_1}{\alpha_1} : \frac{\beta_2}{\alpha_2} = (X,Y,Z,W). \quad \blacksquare$$

Definition 4.21 *A collineation is a spatial transformation, which is a bijection respected to points and lines, and preserves incidence.*

Theorem 4.22 *Any collineation can be represented (in homogeneous coordinates) by the equivalent class of a regular linear transformation given in $\widetilde{A} = \begin{bmatrix} \underline{\underline{A}} & t \\ \underline{u}^T & 1 \end{bmatrix}$ form.*

Definition 4.23 *A collineation is called affinity if $\underline{u} = \underline{0}$. Similarity is an affinity, which preserves the ratio of segments. Isometry is a similarity, which preserves the length of the segments.*

Lemma 4.24 *A collineation is similarity, if $\underline{u} = \underline{0}$ and $\underline{\underline{A}}$ is orthogonal.*

Proof. $\lambda^2 \left\langle \underline{x} - \underline{y}, \underline{x} - \underline{y} \right\rangle = \left\langle \underline{x}' - \underline{y}', \underline{x}' - \underline{y}' \right\rangle = \left\langle \underline{\underline{A}}(\underline{x} - \underline{y}), \underline{\underline{A}}(\underline{x} - \underline{y}) \right\rangle = \left\langle (\underline{x} - \underline{y}), \underline{\underline{A}}^T\underline{\underline{A}}(\underline{x} - \underline{y}) \right\rangle$
But $\lambda^2 \left\langle \underline{x} - \underline{y}, \underline{x} - \underline{y} \right\rangle = \left\langle (\underline{x} - \underline{y}), \lambda^2\underline{\underline{E}}(\underline{x} - \underline{y}) \right\rangle \Rightarrow \underline{\underline{A}}^T\underline{\underline{A}} = \lambda^2\underline{\underline{E}}$. $\blacksquare$

Remark 4.25 *If $\lambda = 1$ then we have an isometry.*

4.3 Spherical geometry

Points: surface of the sphere
Lines: shortest path between two points on the surface of the sphere $\Rightarrow$ great arcs $\Rightarrow$ great circles
Problem: Betweeness is not appropriate

4.3.1 Spherical Order axioms

Axiom Os5 *If $(ACBD)$ then $(BCAD)$ and $(CBDA)$.*

Axiom Os6 *If A, B and D are collinear points then there exists at least one point C such that $(ABCAD)$.*

Axiom Os7 *If four points are situated on a line, there is no more than one 2-2 partition, such that they separate each other.*

Axiom Os8 (Pasch) *Let A, B and C be three points not lying on the same line and e and f be lines, not passing through any of the points A, B, C. Then there exist the points $E \in e$ and $F \in f$ such that either $(AECF)$ or $(BECF)$.*

Remark 4.26 *If f is the ideal line, we get the Euclidean geometry.*

4.3.2 Distance, angle and area

Elliptic geometry: Antipodal points are united: $A = A'$.
Spherical geometry: Every point on the surface of the sphere. $A \neq A'$.
Distance of points: Length of the arc: $r \cdot \alpha = \alpha$ if $r = 1$.
If $\underline{x} = (x, y, z)^T$, then $\underline{x} \sim \underline{\tilde{x}} = (x', y', 1)^T$ if $z \neq 0$.

$$\cos\alpha = \frac{\left\langle \underline{\tilde{a}}, \underline{\tilde{b}} \right\rangle}{\sqrt{\left\langle \underline{\tilde{a}}, \underline{\tilde{a}} \right\rangle \left\langle \underline{\tilde{b}}, \underline{\tilde{b}} \right\rangle}}$$

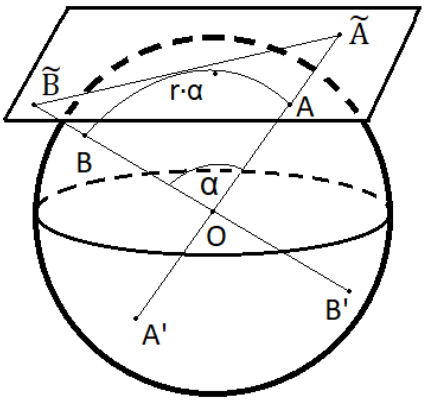


Figure 4.6: Spherical geometry

Lines: Great (half-)circles of the sphere.
Angel of lines: Angles of the orthogonal vectors of the determined planes:

$$\cos\alpha = \frac{\langle \underline{\tilde{n}}_1, \underline{\tilde{n}}_2 \rangle}{\sqrt{\langle \underline{\tilde{n}}_1, \underline{\tilde{n}}_1 \rangle \langle \underline{\tilde{n}}_2, \underline{\tilde{n}}_2 \rangle}}$$

Definition 4.27 *Two lines on the sphere divide the surface into four spherical lune.*

Area of spherical lune:

$$A_{sphere} = 4\pi \Rightarrow A_{lune} = 2\alpha$$

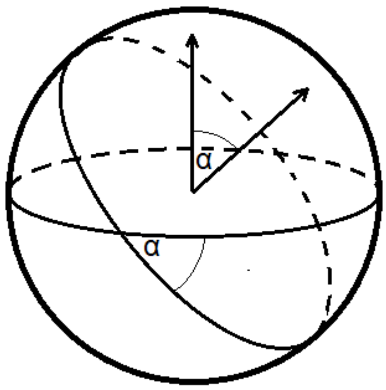


Figure 4.7: Angel of lines, spherical lune

Area of the spherical triangle:
We cover of the sphere with lunes:

$$2(2\alpha + 2\beta + 2\gamma) = 4\pi + 4A_{triangle} \Rightarrow$$

$$A_{triangle} = \alpha + \beta + \gamma - \pi$$

Now, let $\underline{a}$, $\underline{b}$ and $\underline{c}$ be vectors to A, B and C such that they form a right-hand system. Then $(\underline{c} \times \underline{a}, \underline{a} \times \underline{b})\angle = \pi - \alpha$, $(\underline{a} \times \underline{b}, \underline{b} \times \underline{c})\angle = \pi - \beta$, $(\underline{b} \times \underline{c}, \underline{c} \times \underline{a})\angle = \pi - \gamma$.

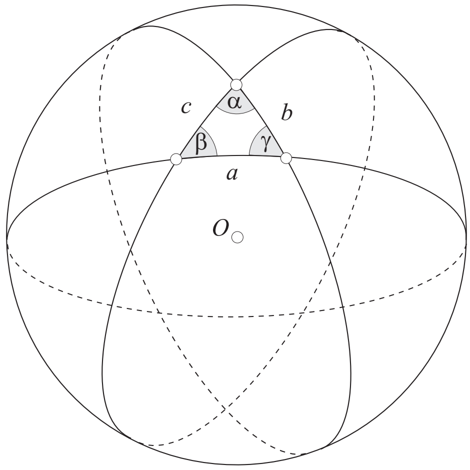


Figure 4.8: Spherical triangle

Theorem 4.28 (Spherical sine theorem) *Let a, b and c be the sides opposite to, and α, β and γ be the angles at the vertices A, B and C of a spherical triangle. Then* $\dfrac{\sin a}{\sin \alpha} = \dfrac{\sin b}{\sin \beta} = \dfrac{\sin c}{\sin \gamma}$.

Proof. According to the Lagrange formula and the definition of the triple product:

$$(\underline{a} \times \underline{b}) \times (\underline{b} \times \underline{c}) = \langle \underline{a}, \underline{b} \times \underline{c} \rangle\, \underline{b} - \langle \underline{b}, (\underline{b} \times \underline{c}) \rangle\, \underline{a} = (\underline{a} \cdot \underline{b} \cdot \underline{c})\underline{b},$$

since $\underline{b} \perp (\underline{b} \times \underline{c})$. This implies that $|(\underline{a} \times \underline{b}) \times (\underline{b} \times \underline{c})| = |\underline{a} \cdot \underline{b} \cdot \underline{c}|$, but $|(\underline{a} \times \underline{b}) \times (\underline{b} \times \underline{c})| = |\underline{a} \times \underline{b}| \cdot |\underline{b} \times \underline{c}| \sin(\pi - \beta) = \sin c \cdot \sin a \cdot \sin \beta$. Similarly, $|(\underline{c} \times \underline{a}) \times (\underline{c} \times \underline{b})| = |\underline{c} \cdot \underline{a} \cdot \underline{b}| = |\underline{a} \cdot \underline{b} \cdot \underline{c}|$ and $|(\underline{c} \times \underline{a}) \times (\underline{c} \times \underline{b})| = \sin b \cdot \sin c \cdot \sin \alpha$. Finally, we obtain that $\dfrac{\sin a}{\sin \alpha} = \dfrac{\sin b}{\sin \beta}$. The other equation can be proved similarly. ■

Theorem 4.29 (Spherical side cosine theorem) *Let a, b and c be the sides opposite to, and α, β and γ be the angles at the vertices A, B and C of a spherical triangle. Then* $\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos \gamma$.

Proof. On the one hand, $\langle (\underline{a} \times \underline{b}), (\underline{b} \times \underline{c}) \rangle = \langle (\underline{a} \times \underline{b}) \times \underline{b}, \underline{c} \rangle = \langle \langle \underline{a}, \underline{b} \rangle\, \underline{b} - \langle \underline{b}, \underline{b} \rangle\, \underline{a}, \underline{c} \rangle = \langle \underline{a}, \underline{b} \rangle \langle \underline{b}, \underline{c} \rangle - \langle \underline{a}, \underline{c} \rangle = \cos c \cdot \cos a - \cos b$. On the other hand, $\langle (\underline{a} \times \underline{b}), (\underline{b} \times \underline{c}) \rangle = |\underline{a} \times \underline{b}| \cdot |\underline{b} \times \underline{c}| \cos(\pi - \beta) = -\sin c \cdot \sin a \cdot \cos \beta \Rightarrow \cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos \beta$. Similarly, $\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos \gamma$ and $\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos \alpha$. ■

Definition 4.30 *Let $ABC\triangle$ be a spherical triangle. Then the $A' := \underline{b} \times \underline{c}$, $B' := \underline{c} \times \underline{a}$ and $C' := \underline{a} \times \underline{b}$ form the polar triangle of $ABC\triangle$ (The distances of the corresponding points are less then $\frac{\pi}{2}$).*

Theorem 4.31 *Let $ABC\triangle$ be a spherical triangle and $A'B'C'\triangle$ be its polar triangle. Then the polar triangle of $A'B'C'\triangle$ is $ABC\triangle$.*

Proof. We know, that the length of the arcs AB', AC', BA', BC', CA' and CB' are equal to $\frac{\pi}{2}$ and the length of the arcs AA', BB' and CC' is less then $\frac{\pi}{2}$, therefore the points A, B and C satisfy the conditions for A'', B'' and C''. ■

Theorem 4.32 *Let $ABC\triangle$ be a spherical triangle with angles α, β and γ, and $A'B'C'\triangle$ be its polar triangle with sides a', b' and c'. Then $a'+\alpha = b'+\beta = c'+\gamma = \pi$, if they are the corresponding sides and angles.*

Proof. $a' = (\underline{c} \times \underline{a}, \underline{a} \times \underline{b})\angle = \pi - \alpha$ ■

Theorem 4.33 (Spherical angle cosine theorem) *Let a, b and c be the sides opposite to, and α, β and γ be the angles at the vertices A, B and C of a spherical triangle. Then* $\cos\gamma = -\cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta \cdot \cos c$.

Proof. We apply the spherical side cosine theorem for the polar triangle $A'B'C'\triangle$: $\cos c' = \cos a' \cdot \cos b' + \sin a' \cdot \sin b' \cdot \cos\gamma' \Rightarrow \cos(\pi-\gamma) = \cos(\pi-\alpha)\cdot\cos(\pi-\beta) + \sin(\pi-\alpha)\cdot\sin(\pi-\beta)\cdot\cos(\pi-c)$. Now applying that $\sin(\pi-\phi) = \sin\phi$ and $\cos(\pi-\phi) = -\cos\phi$ we obtain that $-\cos\gamma = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta\cdot\cos c$. Similarly, $\cos\alpha = -\cos\beta\cdot\cos\gamma + \sin\beta\cdot\sin\gamma\cdot\cos a$ and $\cos\beta = -\cos\alpha\cdot\cos\gamma + \sin\alpha\cdot\sin\gamma\cdot\cos b$. ■

4.4 n-dimensional Euclidean geometry

Definition 4.34 *Let $(\mathbf{V}, \langle\cdot,\cdot\rangle)$ be a vector space with the usual dot product and $\mathbf{E}^n$ be a set. If the ordered point pairs of $\mathbf{E}^n$ can be assigned bijectively to the elements of the vectors space satisfying the following properties, then $\mathbf{E}^n$ is the n-dimensional (analytic) Euclidean space:*

- $\forall P, Q \in \mathbf{E}^n \exists! \underline{v} \in \mathbf{V} : \overrightarrow{PQ} = \underline{v}$
- $\forall P \in \mathbf{E}^n : \overrightarrow{PP} = \underline{0}$
- $\forall P \in \mathbf{E}^n, \underline{v} \in \mathbf{V} \exists! Q \in \mathbf{E}^n : \overrightarrow{PQ} = \underline{v}$
- $\forall P, Q, R \in \mathbf{E}^n : \overrightarrow{PR} + \overrightarrow{RQ} + \overrightarrow{QP} = \underline{0}$

Definition 4.35 *If $\mathbf{V}_k$ is a k-dimensional subspace of $\mathbf{V}$ and $\underline{x}^0 \in \mathbf{E}^n$, then the $\underline{X} = \underline{x}^0 + \mathbf{V}_k$ set is called a k-dimensional affine subspace of $\mathbf{E}^n$. If $k=1$, then it is called line, if $k = n-1$, then it is called hyperplane.*

- $k = 1:\ \underline{x} = \underline{x}^0 + t\cdot\underline{v} \Rightarrow \frac{x_1 - x_1^0}{v_1} = \frac{x_2 - x_2^0}{v_2} = \cdots = \frac{x_n - x_n^0}{v_n}$, and $x_i = x_i^0$ if $v_i = 0$
- $k = n-1:\ \underline{x} = \underline{x}^0 + \sum_{i=1}^{n-1} \alpha_i \underline{v}_i \Rightarrow \{\underline{x} - \underline{x}^0, \underline{v}_1, \ldots, \underline{v}_{n-1}\}$ not linearly independent $\Rightarrow$ $\exists! \underline{n} \neq \underline{0} : \langle \underline{n}, \underline{x} - \underline{x}^0 \rangle = 0 \Rightarrow \langle \underline{x}, \underline{n} \rangle = c$

4.5 Classification of quadratic surfaces

$$(\text{Q}): \quad \underline{x}^T \underline{\underline{A}}\, \underline{x} + 2\underline{b}^T \underline{x} + c = 0$$

is a quadratic form, where $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$, $\underline{b}, \underline{x} \in \mathbb{R}^n$, $c \in \mathbb{R}$ and $\underline{\underline{A}} = \underline{\underline{A}}^T$. By homogeneous coordinates $\underline{X} = \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix}$:

$$(\text{Q}^*): \quad \underline{X}^T \begin{bmatrix} \underline{\underline{A}} & \underline{b} \\ \underline{b}^T & c \end{bmatrix} \underline{X} = 0$$

Now, we change the coordinate system by translating the $\underline{\underline{B}}$ orthogonal system by $\underline{t}$. Then

$$(\text{Q}): \quad \underline{X}^T \begin{bmatrix} \underline{\underline{B}}^T \underline{\underline{A}}\, \underline{\underline{B}} & \underline{\underline{B}}^T (\underline{\underline{A}}\, \underline{t} + \underline{b}) \\ (\underline{t}^T \underline{\underline{A}} + \underline{b}^T) \underline{\underline{B}} & \underline{t}^t \underline{\underline{A}}\, \underline{t} + \underline{b}^T \underline{t} + \underline{t}^T \underline{b} + c \end{bmatrix} \underline{X} = 0$$

$\underline{\underline{B}}$ can be chosen such that $\underline{\underline{B}}^T \underline{\underline{A}}\, \underline{\underline{B}}$ is diagonal and λ_i are the diagonal elements $i = 1 \dots n$.
1. case $\forall i : \lambda_i \neq 0$ $\underline{\underline{A}}$ is invertible and $\underline{t} = -\underline{\underline{A}}^{-1} \underline{b}$.

$$(\text{Q}): \quad \underline{X}^T \begin{bmatrix} \Lambda & \underline{0} \\ \underline{0}^T & -\underline{b}^T \underline{\underline{A}}^{-1} \underline{b} + c \end{bmatrix} \underline{X} = 0,$$

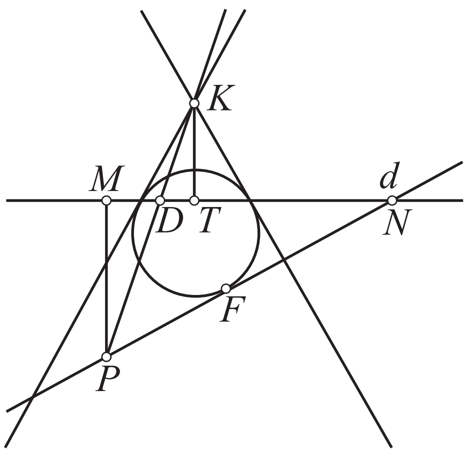
where $\Lambda = diag\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.
2. case $\exists i : \lambda_i = 0$ Let $\underline{s}_i$ be the i^{th} eigenvector such that $\left\langle \underline{s}_i, \underline{s}_j \right\rangle = \delta_{ij}$. Then $\underline{\underline{B}} = \begin{bmatrix} \underline{s}_1 & \underline{s}_2 & \dots & \underline{s}_n \end{bmatrix}$, $\underline{t} = t_1 \underline{s}_1 + t_2 \underline{s}_2 + \dots + t_n \underline{s}_n$ and $\underline{b} = b_1 \underline{s}_1 + b_2 \underline{s}_2 + \dots + b_n \underline{s}_n$. Then $\underline{\underline{A}}\, \underline{t} = \sum_{i=1}^n t_i \underline{\underline{A}}\, \underline{s}_i \Rightarrow (\underline{\underline{B}}^T (\underline{\underline{A}}\, \underline{t} + \underline{b}))_i = t_i \lambda_i + b_i$. If $\lambda_i \neq 0$, then $t_i := -\dfrac{b_i}{\lambda_i}$, otherwise t_i is arbitrary.

$n = 2,\ D = c - \underline{b}^T \underline{\underline{A}}^{-1} \underline{b}$		$D > 0$	$D = 0$	$D < 0$
$\lambda_1 > 0$	$\lambda_2 > 0$	$\emptyset$	$x = y = 0$	ellipse
$\lambda_1 > 0$	$\lambda_2 = 0$	parabola	parabola, $x^2 = 0$	parabola, $\lambda_1 x^2 = -D$
$\lambda_1 > 0$	$\lambda_2 < 0$	hyperbola	$\lambda_1 x^2 + \lambda_2 y^2 = 0$	hyperbola
$\lambda_1 = 0$	$\lambda_2 = 0$	$\emptyset$	$x, y \in \mathbb{R}$	$\emptyset$

$n = 3,\ D = c - \underline{b}^T \underline{\underline{A}}^{-1} \underline{b}$			$D > 0$	$D = 0$	$D < 0$
$\lambda_1 > 0$	$\lambda_2 > 0$	$\lambda_3 > 0$	imaginary ellipsoid	origin	ellipsoid
$\lambda_1 > 0$	$\lambda_2 > 0$	$\lambda_3 = 0$	imaginary elliptic cylinder, elliptic paraboloid	imaginary intersecting planes, elliptic paraboloid	elliptic cylinder elliptic paraboloid
$\lambda_1 > 0$	$\lambda_2 = 0$	$\lambda_3 = 0$	imaginary parallel planes, parabolic cylinder	double plane, parabolic cylinder	parallel planes, parabolic cylinder
$\lambda_1 > 0$	$\lambda_2 < 0$	$\lambda_3 = 0$	hyperbolic paraboloid planes, hyperbolic cylinder	intersecting planes, hyperbolic cylinder	hyperbolic paraboloid, hyperbolic cylinder
$\lambda_1 > 0$	$\lambda_2 > 0$	$\lambda_3 < 0$	two-sheeted hyperboloid	cone	one-sheeted hyperboloid
$\lambda_1 = 0$	$\lambda_2 = 0$	$\lambda_3 = 0$	$\emptyset$	whole space	$\emptyset$

4.6 Conic sections

Definition 4.36 *A double circular cone is formed by a set of lines connecting a common point (apex) to all the points of a circle, where the orthogonal line through the center of the circle to its plane contains the apex. Intersecting a double circular cone by a plane, we obtain conic section.*



One can realize that a sphere can be inscribed by increasing its radius from K and the common points of it with the cone form a circle. Let the plane of this circle be π and our original intersecting plane be α. Let d be the intersection of π and α, the tangent point of the sphere to α be F. Let PK be an arbitrary generating line, where $P \in \alpha$ and let D be the intersection of $\overline{PK}$ and π. Finally, let M and N be the orthogonal projection of P to π and d respectively, and T be the orthogonal projection of K to π.

Since $KTD\triangle \sim PMD\triangle \Rightarrow MPD\angle = \phi$, if ϕ is half of the aperture. $|\overline{PD}| = |\overline{PF}|$, because they are tangent segments to the inscribed sphere $\Rightarrow |\overline{PM}| = |\overline{PD}| \cos\phi$. If ψ is the angle of π and α, then $|\overline{PM}| = \sin\psi|\overline{PN}| = \sin\psi|Pd| \Rightarrow \dfrac{|\overline{PF}|}{|Pd|} = \dfrac{\sin\psi}{\cos\phi} = c$.

Definition 4.37 *We say, that a conic section is ellipse, parabola, hyperbola if this ratio c is smaller than, equal to, greater than 1 respectively.*

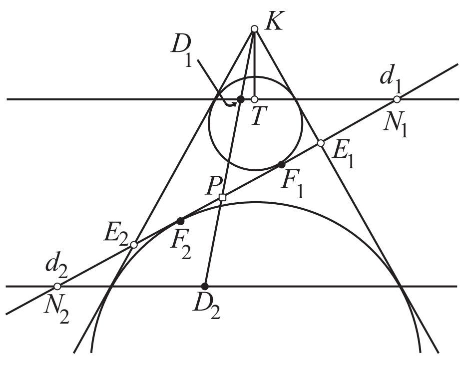


Figure 4.9: Ellipse: $c < 1$

Ellipse: Now, we assume, that α has a common point with every director line ($c < 1$). Then we have two inscribed spheres. Since $|\overline{PD_1}| = |\overline{PF_1}|$ and $|\overline{PD_2}| = |\overline{PF_2}|$, therefore $|\overline{PF_1}| + |\overline{PF_2}| = |\overline{PD_1}| + |\overline{PD_2}| = |\overline{D_1D_2}|$, but $|\overline{D_1D_2}|$ is a segment on a director line $\Rightarrow$ constant $\Rightarrow 2a := |\overline{D_1D_2}|$. For E_i, we can claim, that $|\overline{E_iF_1}| + |\overline{E_iF_2}| = 2a$ (tangent segments). $2c := |\overline{F_1F_2}| \Rightarrow 2a = |\overline{E_1F_1}| + |\overline{E_1F_2}| = 2|\overline{E_1F_1}| + |\overline{F_1F_2}| \Rightarrow |\overline{E_1F_1}| = a - c$. Similarly, $|\overline{E_2F_2}| = a - c \Rightarrow |\overline{E_1E_2}| = 2a$.

Definition 4.38 *Let F_1 and F_2 be two points on a plane such that $d(F_1, F_2) = 2c$ and a be a real number such that $a > c$. Then the locus of points for which the sum of the distances to F_1 and F_2 is $2a$ is called ellipse.*

Parabola: Let α be parallel to exactly one director line ($c = 1$). Then there is no more Dandelin sphere and $|\overline{PF}| = |Pd|$.

Definition 4.39 *Let F be a point and d be a line, not lying on the point. Then the locus of points for which the distance to the point is equal to the distance to the line is equal is called parabola.*

HYPERBOLA: Let α be parallel to exactly one director line ($c > 1$). Then we have two inscribed spheres, one in each part of the double cone. Similarly to the case of the ellipse, $|\overline{D_1D_2}| = |\overline{E_1E_2}| =: 2a$ and $|\overline{F_1F_2}| =: 2c$, but now $c > a$. It is easy to prove, that $\left||\overline{PF_1}| - |\overline{PF_2}|\right| = 2a$

Definition 4.40 *Let F_1 and F_2 be two points on a plane such that $d(F_1, F_2) = 2c$ and a be a real number such that $a < c$. Then the locus of points for which the difference of the distances to F_1 and F_2 is $2a$ is called hyperbola.*

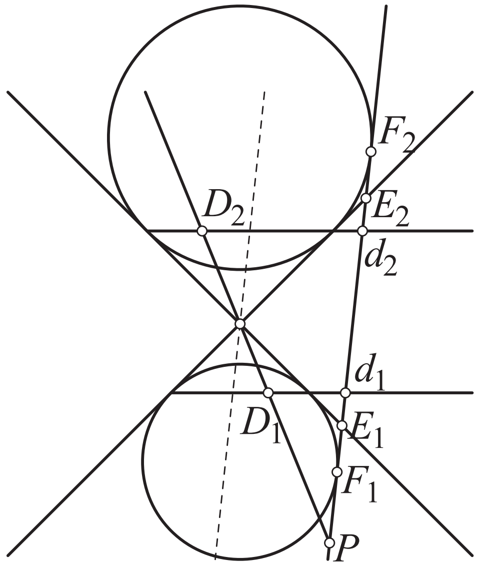


Figure 4.10: Hyperbola: $c > 1$

4.6.1 Tangents from external point to conic sections

ELLIPSE: Let F_1, F_2 be the focis, a be the semi-mayor axis, $b := \sqrt{a^2 - c^2}$ be the semi-minor axis and K be an external point.

Lemma 4.41 *If we draw a circle around F_2 with radius $2a$ then the E_1 reflection of F_1 in any tangent line of the ellipse lies on this circle.*

Proof. Let T be the point of tangency, then $|\overline{F_1T}| = |\overline{E_1T}|$ and the tangent t is the perpendicular bisector of $\overline{E_1F_1}$. Suppose, that T is not on the line of E_1F_2. Then because of the triangle inequality $2a = |\overline{TF_1}|+|\overline{TF_2}| = |\overline{TE_1}|+|\overline{TF_2}| \geq |\overline{E_1F_2}|$. But then $E_1F_2 \cap t \neq \emptyset \Rightarrow Q := E_1F_2 \cap t \Rightarrow 2a \geq |\overline{QE_1}| + |\overline{QF_2}| = |\overline{QF_1}| + |\overline{QF_2}| \Rightarrow Q$ is inside the ellipse and $t = TQ$ is not a tangent, therefore $T = Q$ and $2a = |\overline{TE_1}| + |\overline{TF_2}| = |\overline{E_1F_2}| \Rightarrow E_1$ lies on the circle. ■

Construction. Let t_1 and t_2 be the tangent of the ellipse through K, and the reflection of F_1 in them be E_1 and E_2 respectively. Then $|\overline{KE_1}| = |\overline{KF_1}| = |\overline{KE_2}|$, since t_i is the perpendicular bisector of $\overline{F_1E_i}$. Then E_1 and E_2 lies on the circle around K with radius $|\overline{KF_1}|$ and on the circle around F_2 with radius $2a$. $\Rightarrow t_i$ can be constructed.
■

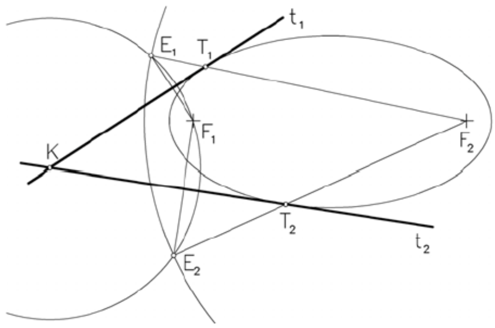


Figure 4.11: Tangents to an ellipse

PARABOLA: Let F be the foci, d be the directrix and K be an external point.

Lemma 4.42 *The E reflection of F in a tangent line lies on the directrix.*

Proof. Let T be the point of tangency, then the directrix is tangential to the circle around T with radius $|\overline{TF}| \Rightarrow d \perp TQ$, where Q is the common point of the circle and the directrix. Then T lies on the perpendicular bisector b of $\overline{FQ}$. Assuming, that b has another common point S with the parabola, we get, that $|\overline{QS}| = |\overline{FS}| = dist(S, d)$, but Q lies on d. Let U be the footpoint of the perpendicular line to d through S, then $UQS\triangle$ is an isosceles triangle with two right angle $\Rightarrow b$ has only one common point T with the parabola $\Rightarrow b$ is the tangent to $T \Rightarrow b = t$ and $E = Q$. ∎

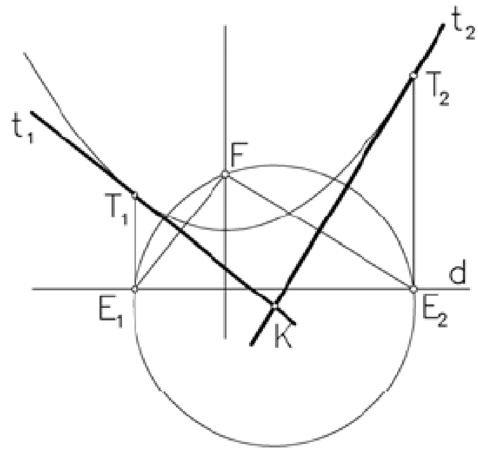


Figure 4.12: Tangents to a parabola

Construction. Let t_1 and t_2 be the tangent of the parabola through K, and the reflection of F in them be E_1 and E_2 respectively. Then $|\overline{KE_1}| = |\overline{KF}| = |\overline{KE_2}|$, since t_i is the perpendicular bisector of $\overline{FE_i}$ and E_i lies on the circle around K with radius $|\overline{KF_1}|$ and on the directrix. $\Rightarrow t_i$ can be constructed. ∎

HYPERBOLA: The construction process is similar to the case of the ellipse.

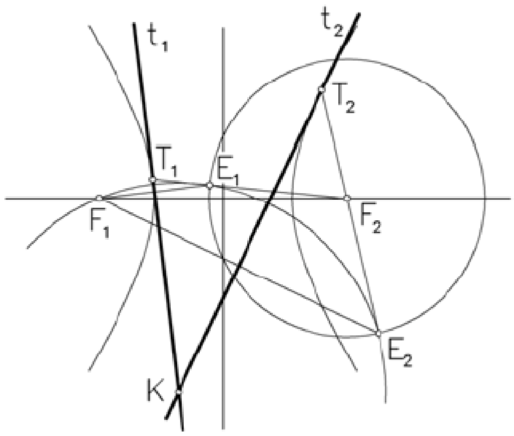


Figure 4.13: Tangents to a hyperbola

4.7 Polyhedrons in n-dimensional Euclidean space

Definition 4.43 *$X = \underline{x}_0 + V_k$ is a k-dimensional affine subspace, if V_k is a real k-dimensional real subspace and $\underline{x}_0 \in \mathbf{E}^n$. If $k = n-1$, X is called hyperplane and can be represented as $\langle \underline{x}, \underline{n} \rangle = c$, where $\underline{n} \in V_{n-1}^{\perp}$. Every hyperplane divides the space into three parts: $H := \{\underline{x} | \langle \underline{x}, \underline{n} \rangle = c\}$, $H^- := \{\underline{x} | \langle \underline{x}, \underline{n} \rangle > c\}$, $H^+ := \{\underline{x} | \langle \underline{x}, \underline{n} \rangle < c\}$.*

Definition 4.44 *Let H be a hyperplane, then $H \cup H^+$ is a closed hyperplane.*

Definition 4.45 *C is an n-dimensional convex convex polyhedron, if it is bounded and can be obtained by the intersection of finitely many closed hyperplanes such that it contains an n-dimensional sphere.*

Definition 4.46 *$H_i \cup H_i^+$ is a crucial hyperplane of C, if $C \cap H_i \neq \emptyset$ and $\exists Q \in C \cap H_i \forall j \neq i : Q \notin H_j$.*

Definition 4.47 *Let C be an n-dimensional convex polyhedron, then the $(n-1)$-dimensional facets of C are the intersections of C with its crucial hyperplanes.*

Lemma 4.48 *The $(n-1)$-dimensional facets of an n-dimensional convex polyhedron are $(n-1)$-dimensional convex polyhedrons.*

Proof. Let Q be a point in C such that H_i is the closest hyperplane to Q and $d := \min_{i \neq j}\{dist(Q, H_j)\}$. Let G be an n-dimensional sphere around Q with radius d, then $G \cap H_i$ is an $(n-1)$-dimensional sphere inside $C \cap H_i$ and $C \cap H_i = \bigcap_{i \neq j}(H_i \cap (H_j \cup H_j^+)) \Rightarrow C \cap H_i$ is an $(n-1)$-dimensional polyhedron . ■

Remark 4.49 *We can construct a chain of polyhedrons. The facets of any k-dimensional polyhedron are $(k-1)$-dimensional polyhedrons.*

Definition 4.50 *The 1-dimensional polyhedrons are edges and the 0-dimensional polyhedrons are vertices.*

Theorem 4.51 *Let C be an n-dimensional convex polyhedron with $\{V_1, V_2, \ldots V_m\}$ vertices. Then C is the convex hull of V_i: $C = \{\sum \alpha_i \underline{v}_i | \alpha_i \geq 0 \wedge \sum \alpha_i = 1\}$.*

Proof. INDUCTION For $k = 0$, it is true and for $k = 1$, we get edges $x = \alpha \underline{v}_i + (1-\alpha)\underline{v}_j$. Now let P be a point on a $(k+1)$-dimensional hyperface. Let v_i be a vertex of this hyperface and $\overrightarrow{V_i P}$ ray intersects it in Q. If Q is a vertex, then P lies on an edge and we are done, otherwise Q lies on a k-dimensional hyperplane $\Rightarrow \underline{q} = \sum_{i=1}^{l} \alpha_i \underline{v}_i$, where $\alpha_i \geq 0$ and $\sum \alpha_i = 1$. Since $P \in \overline{QV_i} : \underline{p} = \alpha \underline{v}_i + \beta \underline{q} = \alpha \underline{v}_i + \beta \sum \alpha_j \underline{v}_j = \sum_{i \neq j} (\alpha_j \beta) \underline{v}_j + (\alpha + \beta \alpha_i) \underline{v}_i$, where $\alpha + \beta = 1 \Rightarrow \alpha + \beta(\sum \alpha_j) = 1$. ■

4.7.1 3-dimensional polyhedrons

Definition 4.52 *S is a star-shaped polyhedron, if any edge belongs to exactly two facets, the facets and the whole surface is simply connected, edge connected, face connected and $\exists P \in S \forall X \in S : \overline{PX} \subset S$.*

Theorem 4.53 (Euler) *Let v, e and f be the number of vertices, edges and facets in a star-shaped polyhedron. Then $v + f = e + 2$.*

Proof. If we project the polyhedron onto a sphere around an appropriate point, then the structure will not change. Now, we chose an inside point of a spherical facet and we apply a stereographic projection to get a planar graph. If there is a circle in it, then erasing an edge of it will result in a planar graph with -1 domain and -1 edge. When there is not any circle in the graph, then we get a tree with v vertices and only 1 domain. This tree has $v-1$ edges $\Rightarrow (v-1)+2 = v+1$. Then $e+2 = v+f$ was true at the beginning. ■

Remark 4.54 *If we are in n-dimension, then $\sum_{i=0}^{n} (-1)^i f_i = 1+(-1)^{n-1}$, where f_i is the number of the i-dimensional facets.*

Definition 4.55 *The union of regularly connected simple polygons is called polyhedral surface if any edge belongs to at most two polygons. The boundary of a polyhedral surface is the union of all the edges which belong to exactly one polygon. A polyhedral surface is closed, if the boundary of it is an empty set.*

Definition 4.56 *A cycle on a polyhedral surface is a closed series of edges, such that every vertex appears exactly twice.*

Lemma 4.57 *A closed polyhedral surface is simply connected, if for every circle, there exists an F set of facets, such that $\forall p \in F, p \notin F \forall p = e_0, e_1, \ldots, e_n = p'$ edge series $\exists i : f_i$ belongs to the cycle.*

Remark 4.58 *The Euler theorem is true for any face connected, simply connected closed polyhedral surface.*

Definition 4.59 *Let C be a convex polyhedron and V be a vertex of it. Then the vertex figure of V is formed by the points, lying on the surface of the unit ball around V and on the edge rays, emanating from V.*

Definition 4.60 *Any C polyhedron is regular, if and only if all of its facets are congruent regular polygons and all of its vertex figures are congruent regular polygons.*

Theorem 4.61 *There exists exactly five regular polyhedron in 3D.*

Proof. Let the facets be regular n-gons and the vertex figures be regular m-gons. Then $2e = f \cdot n = v \cdot m \Rightarrow f = \frac{2e}{n}$ and $v = \frac{2e}{m}$, but according to the Euler theorem: $f + v = \frac{2e}{n} + \frac{2e}{m} = 2 + e \Rightarrow \frac{1}{n} + \frac{1}{m} = \frac{1}{e} + \frac{1}{2} > \frac{1}{2}$ and $n, m \geq 3 \Rightarrow (m, n) \in \{(3,3), (3,4), (4,3), (3,5), (5,3)\}$. Now, we consider the dual of a regular polyhedron as follows. The vertices of the dual polyhedron is the center of the facets, and we connect to vertex if and only if the facets are neighboring.

$(3,3) \Rightarrow$ tetrahedron

$(4,3) \Rightarrow$ cube $\Rightarrow$ its dual is the octahedron

$(5,3) \Rightarrow$ dodecahedron: vertices: $(\pm 1, \pm 1, \pm 1)$, $\left(0, \pm\frac{2}{1+\sqrt{5}}, \pm\frac{1+\sqrt{5}}{2}\right)$, $\left(\pm\frac{2}{1+\sqrt{5}}, \pm\frac{1+\sqrt{5}}{2}, 0\right)$ and $\left(\pm\frac{1+\sqrt{5}}{2}, 0, \pm\frac{2}{1+\sqrt{5}}\right) \Rightarrow$ its dual is the icosahedron. ■

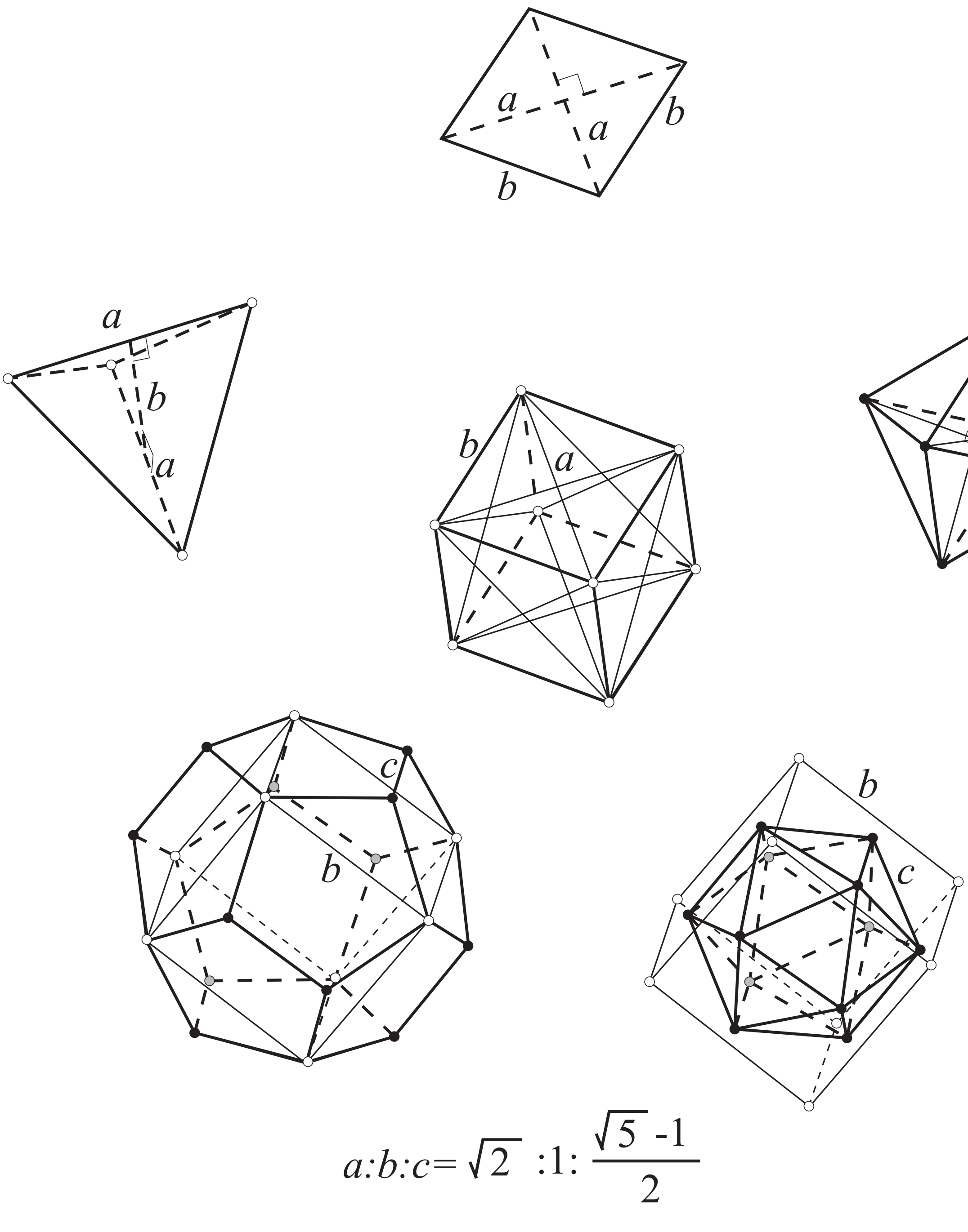


Figure 4.14: Regular polyhedrons

4.8 Congruence of polyhedra

Definition 4.62 *The face lattice of a polyhedron is a partially ordered set, which consists of the vertices, edges and facets, such that order is provided by the set theory containment. Two polyhedra are combinatorically equivalent if their face lattice are isomorphic, i.e, there exists a bijection between vertices, edges and facets such that it preserves order.*

Theorem 4.63 (Cauchy's rigidnes theorem) *If, two convex polyhedra are combinatorically equivalent, and each pair of facets are congruent to each other, then the polyhedrons are congruent.*

Lemma 4.64 (Spherical Arm-lemma) *Let A_i and A'_i be the vertices of two spherical polygon such that $\forall i = 1, 2, \dots, n-1 : \overline{A_iA_{i+1}} \cong \overline{A'_iA'_{i+1}}$ and $\forall i = 2, \dots, n-1 : A_{i-1}A_iA_{i+1}\angle \le A'_{i-1}A'_iA'_{i+1}\angle$. Then $\overline{A_1A_n} \le \overline{A'_1A'_n}$ holds.*

Proof. Induction: For $n = 3$, it is the spherical triangle inequality $\Rightarrow$ true.
Case 1: $\exists i : A_{i-1}A_iA_{i+1}\angle = A'_{i-1}A'_iA'_{i+1}\angle$
Case 2: $\forall i : A_{i-1}A_iA_{i+1}\angle < A'_{i-1}A'_iA'_{i+1}\angle$, but $A_{n-2}A_{n-1}A_n\angle$ can be increased to $A'_{n-2}A'_{n-2}A'_n\angle$, such that $A_1, A_2, \dots, \widetilde{A}_n$ remains convex.
Case 3: $\forall i : A_{i-1}A_iA_{i+1}\angle < A'_{i-1}A'_iA'_{i+1}\angle$, but $A_{n-2}A_{n-1}A_n\angle$ cannot be increased to $A'_{n-2}A'_{n-2}A'_n\angle$, because $A_2A_1\overline{A}_n$ are collinear points, $A_{n-2}A_{n-1}\widetilde{A}_n\angle = A'_{n-2}A'_{n-2}A'_n\angle$, but $A_1, A_2, \dots, \tilde{A}_n$ is non-convex.
Case 1-2: One vertex can be deleted, either A_i or A_{n-1}.
Case 3: We compare $A_2, \dots, A_{n-1}, \overline{A}_n$ $(n-1)$-gon to $A'_2, \dots, A'_n$ $(n-1)$-gon. Then $\overline{A_2\overline{A}_n} \le \overline{A'_2A'_n}$, by induction, but $\overline{A_2A_1} + \overline{A_1\overline{A}_n} = \overline{A_2\overline{A}_n} \le \overline{A'_2A'_n} \le \overline{A'_2A'_1} + \overline{A'_1A'_n}$ and $\overline{A'_2A'_1} = \overline{A_2A_1} \Rightarrow \overline{A_1A_n} \le \overline{A'_1A'_n}$.
■

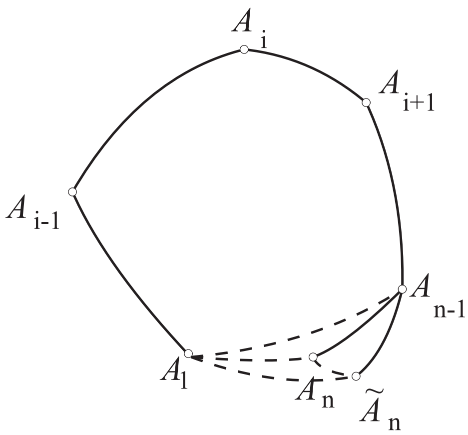


Figure 4.15: Case 2

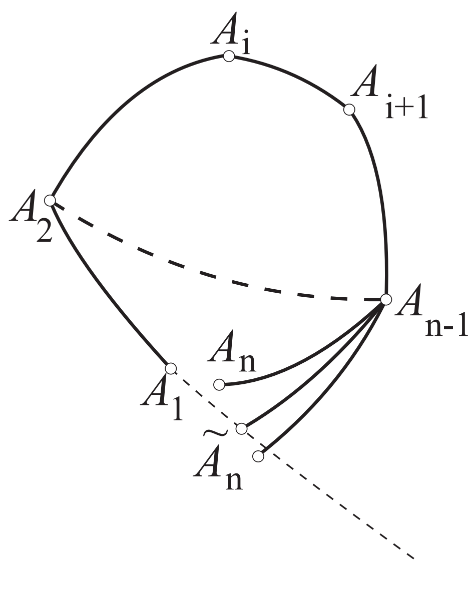


Figure 4.16: Case 3

Assume, that $A_1, \dots, A_n$ and $A'_1, \dots, A'_n$ are spherical polygons, such that $\overline{A_iA_{i+1}} = \overline{A'_iA'_{i+1}}$. If $A_{i-1}A_iA_{i+1}\angle > A'_{i-1}A'_iA'_{i+1}\angle$, then we put $+$ to A_i, if $A_{i-1}A_iA_{i+1}\angle < A'_{i-1}A'_iA'_{i+1}\angle$, then we put $-$ to A_i, otherwise we put 0 to A_i.
There is a sign change, if $A_i = \pm$ and $A_{i+j} = \mp$ if $\forall k = 1, \dots, j-1 : A_{i+k} = 0$.

Lemma 4.65 (Sign lemma) *If, there is a sign on a polygon, then there are at least 4 sign changes.*

Proof. The number of sign changes is even. If, there is a sign, then there ought to be another different sign, otherwise we have a contradiction by our previous lemma. Two sign changes can only be happen if + and − signs are in two blocks. In the two sub-polygons, we apply the spherical arm lemma. We get that the diagonal is smaller and greater in $A'_1, \dots, A'_n$ than in $A_1, \dots, A_n$ at the same time. ■

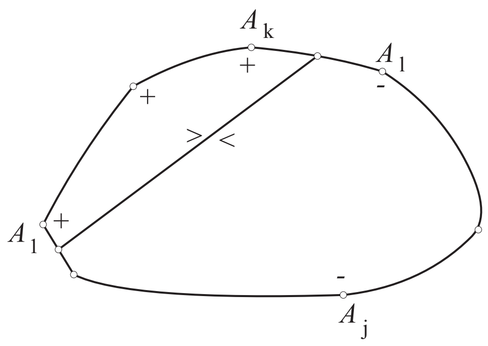


Figure 4.17: Sign lemma

Proof. (Cauchy-theorem)
Case 1: If all the dihedral angles are the same, then we build up the polyhedrons face-by-face.
Case 2: If all the dihedral angles change, then we write + or − if the dihedral angle is greater or smaller in P' than in P. We can apply the sign lemma on the surface of a small sphere around every vertex. Sign change: Two edges, sharing a vertex on a face have different signs $\Rightarrow 4v \leq \delta$, where δ is the number of sign changes. But on the faces $\delta \leq \sum_{k \geq 3} 2 \left[\frac{k}{2}\right] \alpha_k \leq \sum 2(k-2)\alpha_k = 2 \sum k \cdot \alpha_k - 4 \sum \alpha_k = 4e - 4f \Rightarrow 4v \leq \delta \leq 4v - 4f$, but $4v + 4f = 4e + 8$, so this is a contradiction.
Case 3: If some of the dihedral angels change, then a vertex is 'real', if it has an edge with a sign. We delete the ghost edges and the new facets are topologically connected surfaces. Then $v + f \leq e$. We add the ghost edges back one-by one, if one of its endpoint is in the graph. With every possibility, the $v + f \leq e$ inequality remains true $\Rightarrow$ contradiction. ■

5 Area and volume

5.1 Area on the Euclidean plane

Definition 5.1 *Area is an isometry invariant, additive, non-negative set function for simple polygons and 1 is assigned to the unit square.*

Lemma 5.2 *The area of the rectangle is the product of the length of its neighboring sides.*

Lemma 5.3 *The area of the triangle is the half of the product of the length of one of its side and the corresponding altitude.*

Proof. We cut the triangle int two right angled triangle by one of the altitudes. Another copies of these triangles form two rectangles, where the sum of the appropriate sides is the side of the triangle and the other is the corresponding altitude. ■

Lemma 5.4 *Any simple polygon can be divided into triangles by non-intersecting diagonals.*

Proof. INDUCTION: For $n = 3$ it is trivial.
Consider the leftmost point v of the polygon and its neighbors u and v. If $\overline{uv}$ is a diagonal, then we are done. Otherwise, there is at least on vertex of the polygon inside $uvw\triangle$. Consider the furthermost of them to $\overline{uv} \Rightarrow z$. Then $\overline{vz}$ is a diagonal. ■

Remark 5.5 *We get exactly $n - 2$ triangles.*

Definition 5.6 *The area of any simple poygon is the sum of the areas of the triangles, of which the polygon consists.*

Definition 5.7 *A set H has area, if the infimum of the area of the circumscribed polygons is equal to the supremum of the area of the inscribed polygons.*

Lemma 5.8 *Area is a monotonic function, if $A \subseteq B$ and both have area, then $\mathcal{A}(A) \leq \mathcal{A}(B)$.*

Theorem 5.9 *Every convex bounded set has area.*

5.2 Area on the hyperbolic plane

Definition 5.10 *A triangle is called asymptotic, if one of its vertex is a boundary point. A triangle is called doubly/triply asymptotic, if two/three of its vertices are boundary points.*

Theorem 5.11 *All the triply asymptotic triangles are congruent to each other.*

Definition 5.12 *Hyperbolic area is an isometry invariant, additive, non-negative set function for simple polygons and π is assigned to the triply asymptotic triangle.*

Theorem 5.13 *Any asymptotic triangle can be cut off into a pentagon.*

Proof. (Poincaré disk model): Let $ABC\triangle$ be such that C is the ideal point and A is the center of the model. Let D be an ideal point such that (ABD) and M be the footpoint of the perpendicular line to CD through A. Let A_1 be the reflection of B in the line AM and the intersection of BC and A_1D be M_1. Let the footpoints of the perpendicular line to DC through B and A_1 be Q and P respectively. If M_2 is the intersection of A_1P and BC and the reflection of BC in A_1P is DA_2, then $M_2A_1A_2\triangle$ is congruent to $A_1M_1M_2\triangle$ and $M_1M_2B\triangle$. Continuing this procedure, we get the $ABQPA_1$ pentagon. ■

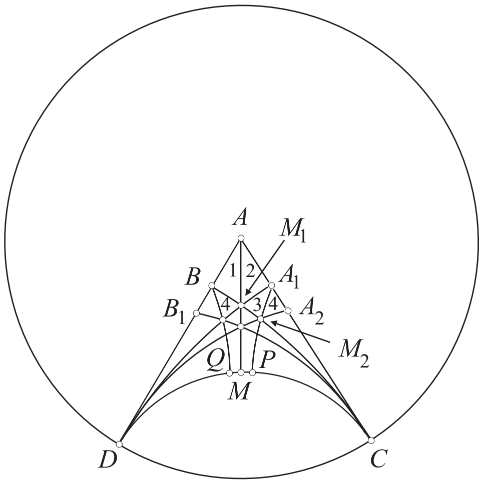


Figure 5.1: Lendin construction

Theorem 5.14 *If, the angle of a doubly asymptotic triangle at the proper vertex is α, then the area of this triangle is $(\pi - \alpha)$.*

Proof. Let $f(\phi)$ be the area, if $\phi = \pi - \alpha$. is the supplementary angle. The union of the two corresponding doubly asymptotic triangle is a triply asymptotic triangle: $\pi = f(\phi) + f(\pi - \phi)$ A triply asymptotic triangle can be cut off into three doubly asymptotic triangle (see Figure 5.2): $\pi = f(\phi) + f(\psi) + f(\pi - \phi - \psi)$. Using the previous result for $\phi + \psi$ we obtain that $f(\phi) + f(\psi) = f(\phi + \psi)$. Our only solution is $f(x) = \lambda x$, since f is monotonously increasing and if $f(1) = \lambda \Rightarrow f(n) = n \cdot \lambda$. Now, if $\frac{k}{n} \leq x \leq \frac{k+1}{n} \Rightarrow k \leq nx \leq k+1 \Rightarrow f(k) \leq f(nx) \leq f(k+1) \Rightarrow \lambda k \leq nf(x) \leq \lambda(k+1) \Rightarrow \frac{k}{n} \leq \frac{f(x)}{\lambda} \leq \frac{k+1}{n} \Rightarrow \forall n : \left| x - \frac{f(x)}{\lambda} \right| \leq \frac{1}{n} \Rightarrow f(x) = \lambda x$. Finally $\pi = f(0) + f(\pi) = f(\pi) \Rightarrow \lambda = 1$. ■

Theorem 5.15 *The area of any hyperbolic triangle is its defect.*

Proof. Let D, E and F be boundary points, such that (ABD), (BCE) and (CAF). Then the area of $DEF\triangle = \pi$ but

$$\pi = area(FDE\triangle) = area(ABC) + area(ADF) + area(BDE) + area(CEF) = area(ABC) + \alpha + \beta + \gamma$$

Therefore $area(ABC) = \pi - (\alpha + \beta + \gamma)$. ■

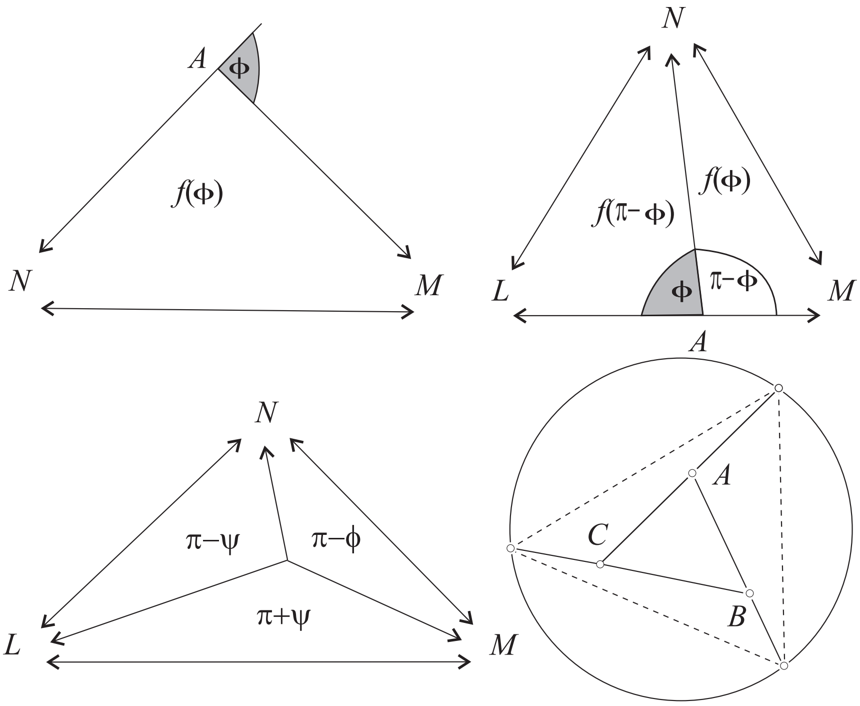


Figure 5.2: Hyperbolic area

5.3 Volume in the Euclidean space

Area: We only used the principum that two polygons have the same area if and only if the first can be cut into finitely many polygonal pieces that can be reassembled to yield the second.

Question: Can we extend this to 3D? (NO!)

Definition 5.16 *Two polyhedrons are scissors-congruent if the first can be cut into finitely many polyhedral pieces that can be reassembled to yield the second.*

DEHN-invariant: We assign a value to every polyhedron such that two scissors-congruent polyhedrons have the same value and $D(\mathcal{P}) = \sum_{i=1}^{n} D(\mathcal{P}_i)$ if $\mathcal{P}$ has been cut into $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$. Let f be an additive function such that $f(0) = f(\pi) = 0$, $l(e)$ be the length of the edge e and Θ_e be the dihedral angle between the two facets meeting at e. Then $D(\mathcal{P}) := \sum_{e \in edges} f(\Theta_e) l(e)$

- If $e \in \mathcal{P}_k$ and e is inside $\mathcal{P}$, then the sum of the dihedral angles around e is $2\pi \Rightarrow f(2\pi)l(e) = 2f(\pi)l(e) = 0$.
- If $e \in \mathcal{P}_k$ and e belongs to a facet of P, then the dihedral angles around e is $\pi \Rightarrow f(\pi)l(e) = 0$.
- If $e \in \mathcal{P}_k$ and e belongs to the e' edge of $\mathcal{P}$, then $f(\Theta_{e'})l(e)$

Theorem 5.17 *The regular tetrahedron $\mathcal{T}$ and the cube $\mathcal{C}$ are not scissors-congruent.*

Proof. Let $l(e) := 1$ for every e edge in $\mathcal{T}$. Then $D(\mathcal{T}) = 6f(\Theta)$, where $\Theta = \arccos\left(\frac{1}{3}\right)$ is the dihedral angle of the regular tetrahedron. It is known, that neither Θ nor π is rational $\exists f$ additive function such that $f(\Theta) = 1$ and $f\left(\frac{\pi}{2}\right) = 0$. Since Θ and π are independent vectors in $\mathbb{R}^1$ over $\mathbb{Q} \Rightarrow \exists$ base with Θ and π. Let $f(\Theta)$ be 1 and $f(b)$ be 0 for all $b \in$base (f is not linear over $\mathbb{R}$). Then f satisfies all conditions and $D(\mathcal{T}) = 6$ but $D(\mathcal{C}) = 0$. ■

Cavalieri principle: The set function V satisfies the Cavalieri principle if the following property is true for any H_1 and H_2 sets in the domain of V: If every element of a parallel plane pencil intersects both H_1 and H_2 in cross-sections of equal are, then $V(H_1) = V(H_2)$.

Definition 5.18 *Volume is an isometry invariant, non-negative, additive set function for simple polyhedrons, which satisfies the Cavalieri principle and 1 is assigned to the unit cube.*

Theorem 5.19 $V(\mathcal{T}) = \frac{1}{3} area(ABC_\triangle) \cdot m$*, where* $m = dist(D, ABC)$.

Proof.

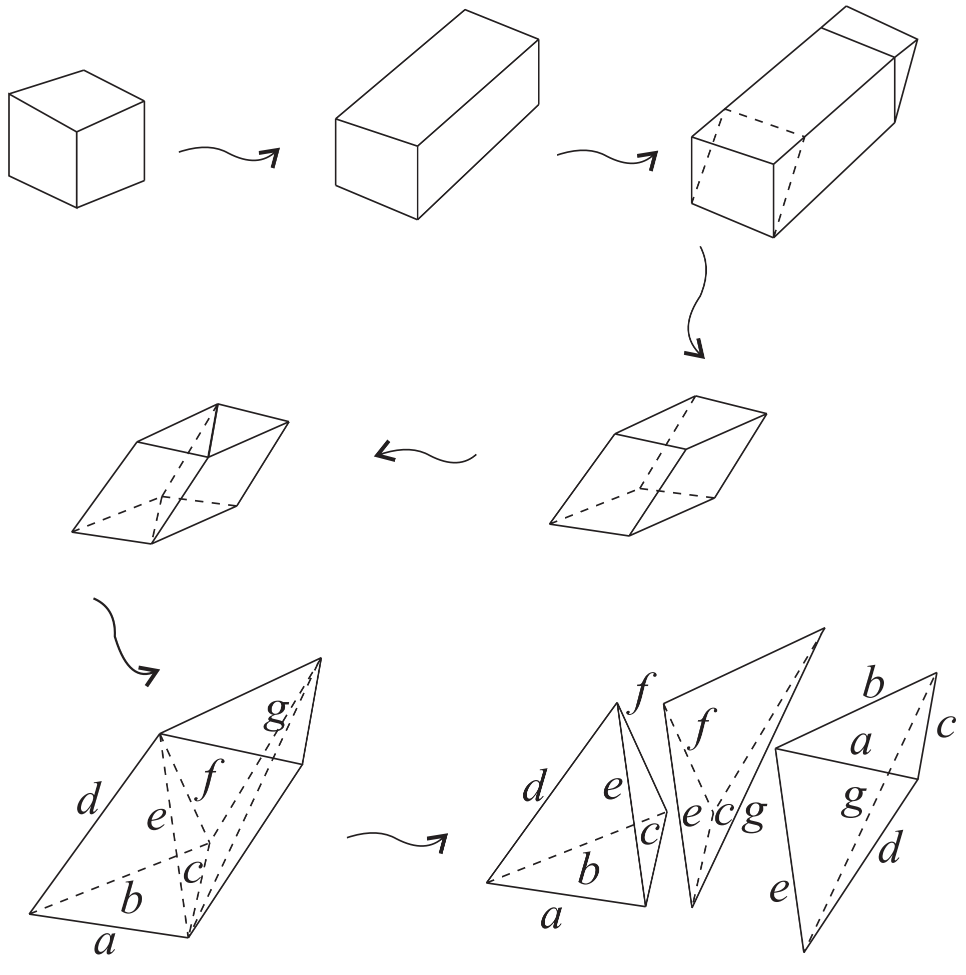


Figure 5.3: Volume of the tetrahedron

■