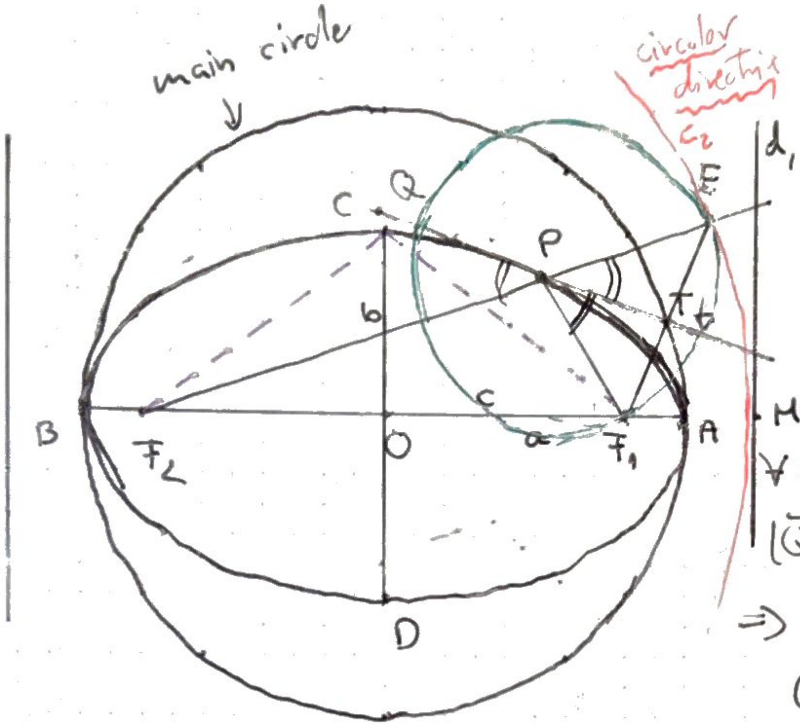
Ellipse $\varepsilon = \frac{c}{a} < 1$



$T := EF \cap t$

$$\left.\begin{array}{l} |\overline{PE}| = |\overline{PF_1}| \\ EPT\sphericalangle = TPF_1\sphericalangle \\ |\overline{PT}| = |\overline{PT}| \end{array}\right\} \; PTE_\triangle \cong PTF_{1\triangle} \Downarrow \; ETP\sphericalangle = F_1TP\sphericalangle = \frac{\pi}{2}, \; |\overline{ET}| = |\overline{F_1T}|$$

$\Rightarrow$ T lies on the perpendicular bisector of $\overline{EF_1}$ $\Rightarrow$

$\forall Q \in t \Rightarrow |\overline{QE}| = |\overline{QF_1}| \Rightarrow$

$|\overline{QF_1}| + |\overline{QF_2}| = |\overline{QE}| + |\overline{QF_2}| \geq |\overline{EF_2}| = 2a$

$\Rightarrow$ Q is an outher point if $Q \neq P$ $\Rightarrow$ t and the ellipse has 1 common point $\Rightarrow$ t is a tangent

Definition by director circle:

Let be given a circle and a point F_1 in it. The center of those circles, that pass through the point and touch the given circle from inside form an ellipse.

Lemma: T lies on the main circle.

Proof: In F_1F_2E triangle OT is a midline $\Rightarrow |\overline{OT}| = \frac{1}{2}\overbrace{|\overline{EF_2}|}^{2a}$

Let $M^{(N_1)}$ be the intersection of the director line d_1 and F_1F_2.

Then we know that $\frac{|\overline{AF_1}|}{|\overline{AM}|} = \frac{c}{a}$ and $(a-c) = |\overline{AF_1}| \Rightarrow |\overline{AM}| = \frac{a}{c}(a-c)$

$|\overline{AM}| = \frac{a^2}{c} - a$ and $|\overline{OA}| = a \Rightarrow |\overline{OM}| = \frac{a^2}{c}$

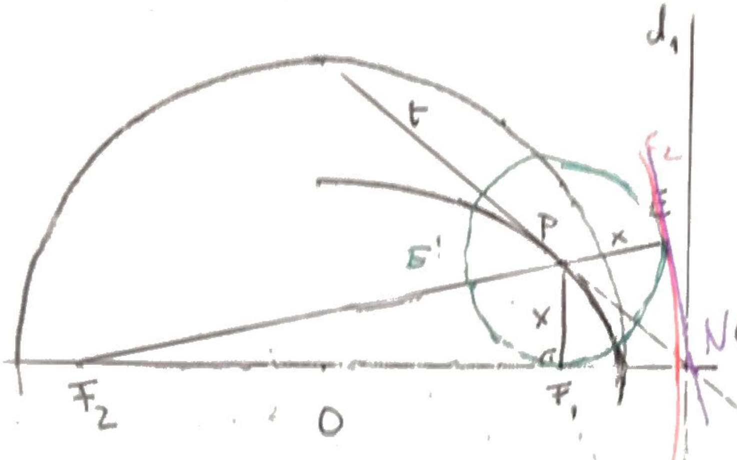
$|\overline{F_2M}| = |\overline{F_2O}| + |\overline{OM}| = c + \frac{a^2}{c} = \frac{a^2+c^2}{c} > 2a$ / $a^2+c^2 > 2ac$, $a^2-2ac+c^2 > 0$, $(a-c)^2 > 0$ /

In OF_1C triangle: $|\overline{OF_1}|^2 + |\overline{OC}|^2 = |\overline{CF_1}|^2$

But $|\overline{CF_1}| + |\overline{CF_2}| = 2a$ and $|\overline{CF_1}| = |\overline{CF_2}| = a$ $\Rightarrow$

$a^2 = b^2 + c^2$

Now, let P be „above' F_1 i.e $F_2F_1P\measuredangle = \frac{\pi}{2}$ and E' be the reflected image of the corresponding E in P.



Notion: PF_1 is called latus rectum (x)

In PF_1F_2 triangle:

$$\left.\begin{array}{l}|PF_2|^2 = |PF_1|^2 + |F_2F_1|^2 = x^2 + (2c)^2 \\ \text{but } |F_2P| = |F_2E| - x = 2a - x\end{array}\right\}$$

$$(2a-x)^2 = x^2 + (2c)^2$$

$$(2a)^2 - 4ax + x^2 = x^2 + (2c)^2$$

$$4a^2 - 4c^2 = 4ax$$

$$x = \frac{a^2-c^2}{a} = \frac{b^2}{a} = a - \frac{c^2}{a}$$

Let the intersection of F_1F_2 and the tangent of c_2 in E be N

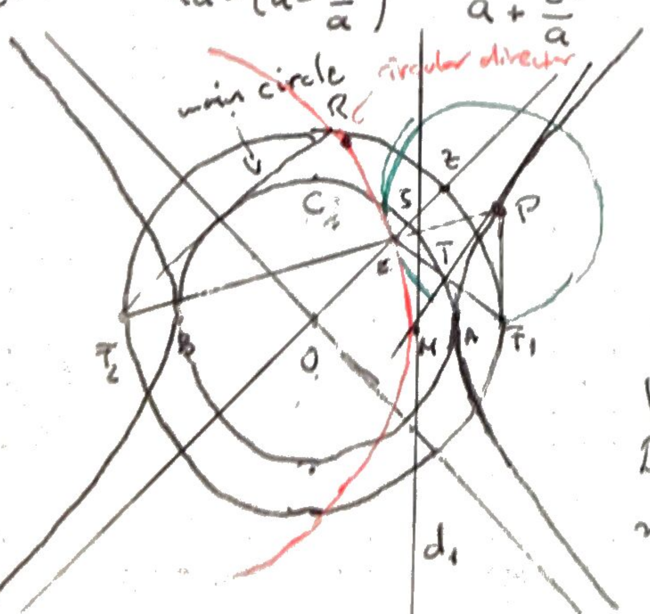
Then $F_1F_2P\triangle \sim EF_2N\triangle \Rightarrow N = M$

$$\frac{|F_2P|}{|F_2F_1|} = \frac{2a-(a-\frac{c^2}{a})}{2c} = \frac{a+\frac{c^2}{a}}{2c} = \frac{a^2+c^2}{2ac} = \frac{\frac{a^2+c^2}{c}}{2a} = \frac{\frac{a^2+c^2}{c}}{|F_2E|} = \frac{|F_2N|}{|F_2E|}$$

but then EN and F_1N are tangents to the director circle $\Rightarrow$

$|F_1M| = |EM| \Rightarrow M \in t$ and $M \in d_1$

$$\frac{|F_2E'|}{|F_2P|} = \frac{2a-2(a-\frac{c^2}{a})}{2a-(a-\frac{c^2}{a})} = \frac{\frac{2c^2}{a}}{a+\frac{c^2}{a}} = \frac{2c^2}{a^2+c^2} = \frac{2c}{\frac{a^2+c^2}{c}} = \frac{|F_2F_1|}{|F_2M|} \Rightarrow E'F_1 \parallel PM$$



Hyerbola $\varepsilon = \frac{c}{a} > 1$

Let S be the tangent point to the main circle from F_1 and the reflection of F_1 in S be R

$|OS| = a$ $|F_1R| = 2|F_1S|$

Dilatation from F_1 by factor 2 results: $O \to F_2$, $S \to R$ $\Rightarrow$

$OS \to F_2R$ and $|F_2R| = 2|OS| = 2a$

Therefore R will be on the circular director, furthermore

$OSF_1\sphericalangle = \frac{\pi}{2} \Rightarrow F_2RF_1\sphericalangle = \frac{\pi}{2}$; $\overline{F_2R} \parallel \overline{OS}$

There is no circle, that touches the circular director in R and passes through F_1 $\Rightarrow$ OS has no common point with the hyperbola $\Rightarrow$ asymptota

OS is also the perpendicular bisector of F_1R

Let M be the footpoint of the perpendicular line to F_1F_2 through S, then $OSM_\Delta \sim OSF_1 \Rightarrow \frac{|\overline{OS}|}{|\overline{OM}|} = \frac{|\overline{OF_1}|}{|\overline{OS}|} \Rightarrow$

$$|\overline{OM}| = \frac{|\overline{OS}|^2}{|\overline{OF_1}|} = \frac{a^2}{c} \Rightarrow |\overline{MA}| = |\overline{OA}| - |\overline{OM}| = a - \frac{a^2}{c} = \frac{ac-a^2}{c} = \frac{a(c-a)}{c}$$

But $|\overline{AF_1}| = c-a \Leftrightarrow \frac{|\overline{AF_1}|}{|\overline{AM}|} = \frac{c}{a} \Rightarrow$ M lies on the director line.

$$|\overline{MF_1}| = |\overline{OF_1}| - |\overline{OM}| = c - \frac{a^2}{c} = \frac{c^2-a^2}{c} := \frac{b^2}{c}$$

Now let P be „above" F_1, i.e. $F_2F_1P\sphericalangle = \frac{\pi}{2}$, then

$$|\overline{F_1F_2}|^2 + |\overline{F_1P}|^2 = |\overline{F_2P}|^2 = (|\overline{F_2E}| + |\overline{EP}|)^2 = (|\overline{F_2E}| + |\overline{F_1P}|)^2$$

$$(2c)^2 + x^2 = (2a+x)^2$$

$$4c^2 + x^2 = 4a^2 + 4ax + x^2 \Rightarrow x = \frac{c^2-a^2}{a} = \frac{b^2}{a}$$ $\swarrow$ latus rectum

Let the intersection of the tagent at P and F_1F_2 be N.

Then $F_2PN\sphericalangle = F_1PN\sphericalangle$ (E lies on F_2P and $EPT_\Delta \cong F_1PT_\Delta$)

$$\Rightarrow \frac{|\overline{F_2N}|}{|\overline{F_1N}|} = \frac{|\overline{PF_2}|}{|\overline{PF_1}|} = \frac{2a + \frac{b^2}{a}}{\frac{b^2}{a}} = \frac{2a^2+b^2}{b^2} = \frac{2a^2+c^2-a^2}{c^2-a^2} = \frac{a^2+c^2}{a^2-a^2}, \text{ but}$$

$$\frac{|\overline{F_2M}|}{|\overline{F_1M}|} = \frac{|\overline{F_2F_1}| - |\overline{F_1M}|}{|\overline{F_1M}|} = \frac{2c - \frac{b^2}{c}}{\frac{b^2}{c}} = \frac{2c^2-b^2}{b^2} = \frac{2c^2-c^2+a^2}{c^2-a^2} = \frac{a^2+c^2}{c^2-a^2} \;\;)) \Rightarrow N = M$$

$$\frac{|\overline{TM}|}{|\overline{TF_1}|} = \frac{|\overline{MF_1}|}{|\overline{PF_1}|} = \frac{\frac{b^2}{c}}{\frac{b^2}{a}} = \frac{a}{c} = \frac{|\overline{OC}|}{|\overline{OF_1}|} \Rightarrow OF_1C_\Delta \sim TF_1M_\Delta \Rightarrow$$

$$\Rightarrow MF_1T\sphericalangle \cong CF_1O\sphericalangle \Rightarrow C -\!\!\bullet\!\!- TF_1 = EF_1$$