Quadratic forms

1.Def: The $a_{11}x_1^2 + a_{22}x_2^2 + a_{33} + 2a_{12}x_1x_2 + 2a_{13}x_1 + 2a_{23}x_2 = 0$ equations is a quadratic form.

Alternative form: $\underline{x}^T \underline{\underline{A}}\,\underline{x} + 2\underline{b}^T\underline{x} + \underline{c} = 0$, where $\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ $\underline{b} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$ $c = a_{33}$

and $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Reminder: Homogeneous coordinates: $[\underline{x}] \rightsquigarrow \begin{bmatrix} \underline{x} \\ 1 \end{bmatrix}$

Def: The homogeneous coordinates of a ~~spatia~~ point P is the equivalence class of the assigned point

e.g.: $P(p) - \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightsquigarrow \tilde{p} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \sim \left\{ \alpha \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \setminus \{0\} \right\}$

Q is ideal, if $Q \sim \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$ $\quad \nexists R \sim \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Homogeneous form of the quadratic form:

$\tilde{\underline{x}}^T \begin{bmatrix} \underline{\underline{A}} & \underline{b} \\ \underline{b}^T & c \end{bmatrix} \tilde{\underline{x}} = 0$ $\qquad [x_1 \; x_2 \; x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

1st question: where is the center of the conic (if it exists?)

2nd question: What are the axis?

2.Theorem: Let us be given a ~~conic with in the~~ its homogeneous quadratic form form $\tilde{\underline{x}}^T \hat{\underline{\underline{A}}} \tilde{\underline{x}} = 0$. Then its center can be determined by the solution of the $\left.\begin{matrix} a_{11}x + a_{12}y + a_{13} = 0 \\ a_{21}x + a_{22}y + a_{23} = 0 \end{matrix}\right\}$ equation system.

Proof: The $(x_1 - x) = x_1'$ and $(x_2 - y) = y_1'$ substitution will transform our quadratic equation to a centered quadratic form

$a_{11}(x_1' + x)^2 + a_{22}(x_2' + y)^2 + 2a_{12}(x_1' + x)(x_2' + y) + 2a_{13}(x_1' + x) + 2a_{23}(x_2' + y) + a_{33} = 0$

where x and y are such that the coefficient of x_1' and x_2' is 0

$x_1': \; + 2a_{11}x + 2a_{12}y + 2a_{13} = 0$ $\qquad$ Solution? $\square$ $\qquad$ Later!

$x_2': \; 2a_{22}y + 2a_{12}x + 2a_{23} = 0$

3.Lemma: Conic sections are quadratic forms.

Proof: All the canonical forms are quadratics.

4 Def: Consider the homogeneous form of the quadratic form.
It is called elliptic/paraboitic/hyperbolic if the number of ideal points, satisfying the homogeneous form is 0/1/2 respectively.

5 Theorem: The quadratic form is elliptic/parabolic/hyperbolic if $A_{33} > 0 / = 0 / < 0$.

Proof: Q is ideal if $[x_1\ x_2\ 0]$. ~~Replacing the equation with~~

$\Rightarrow \quad a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = 0$

$x_1, x_2 = 0$ is a good solution, but not accepted.

If $[x_1\ x_2\ 0]$ is a good solution, then $[\alpha x_1\ \alpha x_2\ 0]$ is also a good but considered to be the same solution (in hom. sens)

(*)

1) If $a_{11} = a_{22} = 0 \Rightarrow a_{12} \neq 0 \Rightarrow x_1x_2 = 0 \Rightarrow$ 2 solutions $[1\ 0]$ $[0\ 1]$.
but then $A_{33} = -a_{12}^2 < 0$ ✓

2) If (e.g.) $a_{11} \neq 0 \Rightarrow x_2 \neq 0$ (replacing 0 to x_2: $a_{11}x_1^2 = 0 \Rightarrow x_1 = 0$ $[0,0]$ ↯)

$\Rightarrow a_{11}\left(\frac{x_1}{x_2}\right)^2 + 2a_{12}\left(\frac{x_1}{x_2}\right) + a_{22} = 0$

(*) $\Rightarrow$ Two solutions are different, if $\frac{x_1}{x_2}$ $(x_2 \neq 0)$ ratio is different ↯

$\left(\frac{x_1}{x_2}\right)_{1,2} = \ldots \qquad D = 4a_{12}^2 - 4a_{11}a_{22} = 4A_{33}$

$D > 0 \Rightarrow A_{33} < 0$, 2 sol

$D = 0 \Rightarrow A_{33} = 0$ 1 sol

$D < 0 \Rightarrow A_{33} > 0$ 0 sol

6 Def The $x = [x_1\ x_2\ x_3]$ point is conjugated to $y = [y_1\ y_2\ y_3]$ respected to the Q quadratic curve, if $\underbrace{[x_1\ x_2\ x_3]\, Q \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{x^TQy} = \sum a_{ij} x_i y_j = 0$.

7 Lemma: x is conjugated to x respected to Q if $x \in Q$. □

8 Lemma: The locus of point $\underline{y}$ conjugated to $\underline{x}$ form a line or a plane(!)

Proof: $0 = \sum a_{ij} x_i y_j = (\underbrace{a_{11}x_1 + a_{12}x_2 + a_{13}x_3}_{u_1})y_1 + (\underbrace{a_{12}x_1 + a_{22}x_2 + a_{23}x_3}_{u_2})y_2 + (\underbrace{a_{13}x_1 + a_{23}x_2 + a_{33}x_3}_{u_3})y_3$

If all $u_i = 0 \Rightarrow y$ can be anything (plane) otherwise $u_1y_1 + u_2y_2 + u_3y_3 = 0$ form a line.

9 Def: If the conjugates ~~for~~ to $\underline{x}$ form a line, then it is called the polar line of $\underline{x}$. If $\underline{x}$ ~~it~~ is conjugated to the whole plane then $\underline{x}$ is called singular. A quadratic curve is called singular if it has at least 1 singular point, otherwise it is called regular. #

10 Lemma: A qudratic curve is singular iff $\det A = 0$.

~~B~~ Proof: According to the previous proof:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= u_1 = 0 \\ a_{12}x_1 + a_{22}x_2 + a_{23}x_3 &= u_2 = 0 \\ a_{13}x_1 + a_{23}x_2 + a_{33}x_3 &= u_3 = 0 \end{aligned}\right\}$$ has a nontrivial solution for $\underline{x} \Leftrightarrow \det A = 0$

11 Theorem: A singular quadratic curve is either a single point or a single line or an intersecting double line.

Proof ~~(sketch)~~:

Lemma: If $\underline{x}, \underline{y} \in Q$ ($\underline{x}^T\underline{\underline{A}}\underline{x}=0$ and $\underline{y}^T\underline{\underline{A}}\underline{y}=0$) are conjugated to each other, then the whole line, determined by $\underline{x}, \underline{y}$ belongs to Q.

$\underline{x}\underline{\underline{A}}\underline{y} = 0 \Rightarrow \lambda\underline{x} + \overset{\mu}{\cancel{(1-\lambda)}}\underline{y} \in Q$ $\quad (\lambda\underline{x} + \overset{\mu}{\cancel{(1-\lambda)}}\underline{y})^T\underline{\underline{A}}(\lambda\underline{x} + \overset{\mu}{\cancel{(1-\lambda)}}\underline{y}) =$

$= \lambda^2\underline{x}^T\underline{\underline{A}}\underline{x} + 2\overset{\mu}{\cancel{(1-\lambda)}}\lambda\,\underline{y}^T\underline{\underline{A}}\underline{x} + \overset{\mu}{\cancel{(1-\lambda)}}{}^2\underline{y}^T\underline{\underline{A}}\underline{y} = 0$ $\square$ L.

1) If S is singular $\Rightarrow$ S is conjugate to the whole plane $\Rightarrow$ S is conjugate to S $\Rightarrow S \in Q \Rightarrow Q$ is not empty.

2) If ~~...~~ $S \neq A \in Q \Rightarrow S$ is conjugate to $A \overset{\text{Lemma}}{\Rightarrow} SA \in Q$
$\Rightarrow Q$ consists of lines through S

3) Any line, not containing S cannot have more than 2 intersections with ~~the~~ Q. ~~Otherwise $\exists\lambda\mu$~~ $\underline{x}_1$ if $\underline{x}_1, \underline{x}_2, \underline{x}_3 \in Q \Rightarrow \exists\lambda \neq 0, 1$
$\underline{x}_3 = \lambda\underline{x}_1 + (1-\lambda)\underline{x}_2 \Rightarrow 0 = \underline{x}_3^T\underline{\underline{A}}\underline{x}_3 = \dots = \lambda^2\underbrace{\underline{x}_1^T A\underline{x}_1}_{0} + (1-\lambda)^2\underbrace{\underline{x}_2^T\underline{A}\underline{x}_2}_{0} + 2\lambda(1-\lambda)\underline{x}_1^T\underline{A}\underline{x}_2$
$\Rightarrow \underline{x}_1$ is conjugate to $\underline{x}_2 \Rightarrow x_1x_2$ line is on $Q \Rightarrow$ any point on this line with S is on $Q \Rightarrow Q$ is the whole plane ↯.

(#) If there is exactly ~~a~~ 1 point that is conjugate to all the points of a line then this point is called the pole of the line.

Polarity to regular conic sections

12 Theorem: For a regular conic every 1) point has a polar and every line 2) has a pole.

Proof: Memory: Q is regular => not singular => $\nexists$ x conjugate to the whole plane => every point is conjugate to all points of a line 1) ✓

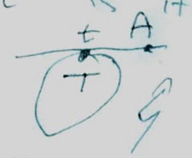
2) For the previous line is conjugate to that point, it is only necessary to prove that this point is unique. But this polarity is related to the

pole
polar

$$(a_{11}x_1 + a_{12}x_2 + a_{13})y_1 + (a_{12}x_1 + a_{22}x_2 + a_{23}x_3)y_2 + (a_{13}x_1 + a_{23}x_2 + a_{33}x_3)y_3 = 0$$

$$\Rightarrow \left.\begin{matrix} a_{11}x_1 + \dots = u_1 \\ a_{12}x_1 + \dots = u_2 \\ a_{13}x_1 + \dots = u_3 \end{matrix}\right\}$$ equation. But this has unique solution. ☒

13 Theorem: The pole of a tangent to regular conic is their common point, and the polar of any point of a regular conic is its tangent.

Proof: a)


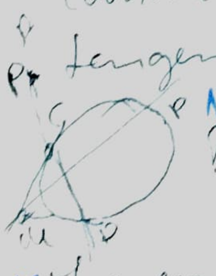
T is conjugated to T $T \sim T$

↯ if A lies on the conic, then $A \sim A$ =>

TA lies on the conic => Q is singular ↯

14 Theorem: If a polar of a point has two common points with a conic, then ~~it is the~~ they are the tangent points drawn to the conic from the given point.

15 Theorem: Considering any line through an other point P respected to a conic such that the number of the common points is 2 then the intersection of the tangents drawn to these points lie on the polar of P.



16 Def: A point is a center to the conic if its polar is ideal.

17 Theorem: A conic has center iff $A_{33} \neq 0$

Proof $[x_1\ x_2\ x_3] \sim [y_1\ y_2\ 0] \iff y_1(a_{11}x_1 + a_{12}x_2 + a_{13}x_3) = 0$

$\forall y_1, y_2: y_2(a_{21}x_1\ a_{22}x_2 + a_{23}x_3) = 0$

$x_1 : x_2 : x_3 = A_{31} : A_{32} : A_{33}$

$x_3 = 0 \iff A_{33} = 0$

18 Cor The center of a centered conic is $(0,0)$ iff $a_{13}=a_{23}=0$.

19 Def: The diameter of a regular conic are the lines of which pole is ideal.

20 Cor The diameters of a centered conic are the lines through the center, the diameters of the parabolo are the lines,
~~Def:~~ parallel to its axis (orthogonal lines to the directrix).

21 Remark: The tangent(s) drawn to the common point(s) of a centered conic to any of its diameter are parallel (their intersections are on the pole that is ideal)

22 Def ~~Theorem~~ $\Rightarrow$ conjugated direction
If this conjugated direction is orthogonal to the diameter, then it is the axis of the conic.

23 Theorem: The ~~axis of a conic~~ d The conjugated direction of an axis of a conic is $[x_1 x_2]$ if and only if

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 &= \lambda x_2 \end{aligned}\right\}$$

Proof: The polar of $[x_1\ x_2\ 0]$ ideal point:

$a_{11}x_1 + a_{12}x_2 : a_{21}x_1 + a_{22}x_2 : a_{31}x_1 + a_{32}x_2$. This line is common iff ~~not~~ same all of the first 2 ~~coor~~ value is 0 and

Then its normal is $\underline{n} = (a_{11}x_1 + a_{12}x_2;\ a_{21}x_1 + a_{22}x_2) \Rightarrow$ is an axis if it is orthogonal to (x_1, x_2).

$\underline{n} \parallel (x_1; x_2) \Rightarrow \square$.

24 Theorem If $A = A^T$ then it has either 2 orthogonal eigendirection or all direction is an eigendirection.

Proof: $\det(A-\lambda I)$ is a quadratic to λ $D = (a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}^2)$
$= (a_{11}-a_{22})^2 + 4a_{12}^2 \geq 0$

1) $D = 0 \Leftrightarrow a_{12} = 0 \wedge a_{11} = a_{22}$: $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ $\begin{aligned} x' &= ax \\ y' &= ay \end{aligned}$ $\forall (x,y)$ /circle/

2) $D > 0$ $\lambda_1 \neq \lambda_2$ ...

$\square$

Theorem: Every quadratic curve can be transformed to one of the following form:

a) $\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = \begin{cases} 1 & \text{ellipse or circle } (a=b) \\ -1 & \text{empty curve (ideal ellipse or circle)} \\ 0 & \text{point} \end{cases}$ elliptic

b) $\frac{\xi^2}{a^2} - \frac{\eta^2}{b^2} = \begin{cases} \pm 1 & \text{hyperbola} \\ 0 & \text{intersecting lines} \end{cases}$ hyperbolic

c) $\eta^2 = 2p\xi$ parabola

$\frac{\xi^2}{a^2} = \begin{cases} 1 & \text{parallel lines} \\ 0 & \text{(double) line} \\ -1 & \text{empty curve (ideal double line)} \end{cases}$ parabolic

Proof: 1) $A_{33} \neq 0 \Rightarrow \lambda_1 \neq \lambda_2$ $\lambda_1 \lambda_2 \neq 0 \Rightarrow$ central conic

First we apply the $x' = x - c_x$ $y' = y - c_y$ substitution to eliminate a_{13} and a_{23}' coefficiants. Then the new quadratic will be $a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + a_{33}' = 0$

The eigenvectors of λ_1 and λ_2 are $\underline{s} = \binom{s_1}{s_2}$ and $\underline{t} = \binom{t_1}{t_2}$ $|\underline{s}| = 1$ $|\underline{t}| = 1$ s.t.

~~Then the~~ $\cancel{x_1' = s_1 x_1' + t_1 x_2'}$ $\cancel{\eta = s_2 x_1' + t_2 x_2'}$ ~~substitution~~

~~$a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 = (a_{11}x_1' + a_{12}x_2')x_1 + (a_{21}x_1 + a_{22}x_2)x_2 =$~~

Then $\xi = s_1 x_1' + s_2 x_2'$ $\iff$ $x_1' = s_1\xi + t_1\eta$

$\eta = t_1 x_1 + t_2 x_2'$ $\quad$ $x_2' = s_2\xi + t_2\eta$ $\Rightarrow \lambda_1\xi^2 + \lambda_2\eta^2 + a_{33} = 0$

a) $0 < A_{33} = \lambda_1 \lambda_2 \Rightarrow \operatorname{sign}\lambda_1 = \operatorname{sign}\lambda_2$

If $a_{33} < 0 \overset{\lambda_i > 0}{\Rightarrow} \lambda_1\xi^2 + \lambda_2\eta^2 = 0$ $/ : \lambda_1\lambda_2$ $\nearrow$ $\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = \begin{matrix} \pm 1 \\ 0 \end{matrix}$

b) $0 > A_{33} = \lambda_1 \lambda_2 \Rightarrow \operatorname{sign}\lambda_1 = -\operatorname{sign}\lambda_2 \Rightarrow \frac{\xi^2}{a^2} - \frac{\eta^2}{b^2} = \begin{matrix} \pm 1 \\ 0 \end{matrix}$

2) $A_{33} = 0 = \lambda_1 \lambda_2 \rightarrow \lambda_1 = 0$ $\lambda_2 \neq 0$ $\quad x_1' = s_1 x + t_1 y$

$x_2' = s_2 x + t_2 y$

$Q \rightsquigarrow \lambda_2 y'^2 + 2a_{13}'x' + 2a_{23}'y' + a_{33}' = 0 \longrightarrow \eta = y + \frac{a_{23}}{\lambda_2} \rightsquigarrow$

$\lambda_2\eta^2 + 2a_{13}x + d = 0 \rightsquigarrow \xi = x + \frac{d}{2a_{13}}$ $\overset{(1)}{\underset{a_{13} \neq 0}{\rightsquigarrow}} \lambda_2\eta^2 + 2a_{13}\xi = 0 \rightarrow \eta^2 = -\frac{2a_{13}}{\lambda_2}\xi$ $\quad 2p$

(2) $a_{13} = 0$: $\eta^2 = \frac{d}{\lambda_2}$ $\quad \frac{\eta^2}{a^2} = \pm 1 / 0$