Hyperbolic geometry

1.1 Parallel lines on the hyperbolic plane

Let TA be perpendicular to a such that T lies on a. Furthermore let $\overrightarrow{AK}$ be perpendicular to AT $\Rightarrow AK \cap a = \emptyset$. Consider the rays emanating from A in $TAK\angle$. They intersect the segment $\overline{TK}$ for Lemma 1.17. Assign a ray to the set $\mathcal{A}$ if it intersects a and to $\mathcal{B}$ if not. Then, apply the Dedekind axiom to these sets $\Rightarrow$ *limiting parallels*.

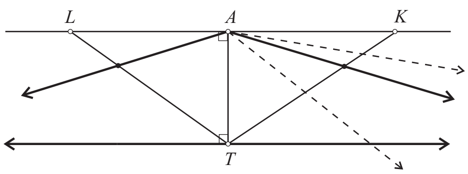


Figure 1.1: Limiting parallels

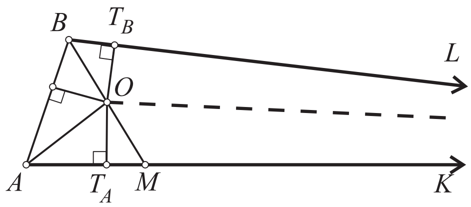
Definition 1.1 *$\overrightarrow{AC} \parallel \overrightarrow{TR}$, if $\overrightarrow{AC}$ is the limiting parallel to $\overrightarrow{TR}$. If $D \in \mathcal{A}$, then $\overrightarrow{AD}$ called intersecting line of $\overrightarrow{TR}$. If $E \in \mathcal{B}$, then $\overrightarrow{AE}$ called ultraparallel line to $\overrightarrow{TR}$.*

Lemma 1.2 *If $\overrightarrow{AB} \parallel \overrightarrow{CD}$, then for any point C^* on the line of CD: $\overrightarrow{AB} \parallel \overrightarrow{C^*D}$.*

Proof. One can distinguish 2 cases, both can be proved by the Pasch axiom. ■

Theorem 1.3 *The parallelism of rays is symmetric and transitive.*

Proof. Symmetry: Let $\overrightarrow{AK} \parallel \overrightarrow{BL}$ be such that L and K are points "far enough". Draw the angle bisector of $BAK\angle$ and $ABL\angle$, they intersect each other in O. Let T_A and T_B be points on $\overrightarrow{AK}$ and $\overrightarrow{BL}$ respectively, such that $OT_A \perp AK$ and $OT_B \perp BL$. Then $\overrightarrow{T_AK} \parallel \overrightarrow{T_BL}$ for the previous lemma. But then $\overrightarrow{T_BL} \parallel \overrightarrow{T_AK}$, since it is symmetric to the angle bisector of $T_BOT_A\angle$.



Transitivity (Only in the plane and only for one case): $\overrightarrow{AK} \parallel \overrightarrow{BL} \wedge \overrightarrow{BL} \parallel \overrightarrow{CM} \overset{?}{\Rightarrow} \overrightarrow{AK} \parallel \overrightarrow{CM}$

1) If $\overrightarrow{CM} \cap \overrightarrow{AK} \neq \emptyset \underbrace{\Rightarrow}_{\text{Pasch}} \overrightarrow{CM} \cap \overrightarrow{BL} \neq \emptyset$.

2) If f is a ray, emanating from C in $BCM\angle \Rightarrow f \cap \overrightarrow{BL} \neq \emptyset \Rightarrow B'$ and $\overrightarrow{B'L} \parallel \overrightarrow{AK} \Rightarrow f$ also intersects $\overrightarrow{AK}$ ■

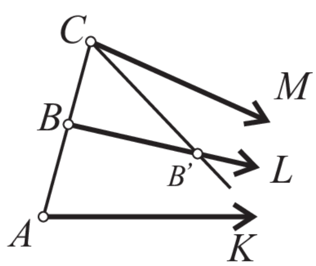


Figure 1.2: Proof: Theorem 1.3

Definition 1.4 *Two line is parallel to each other if they have parallel rays.*

1.1.1 Perpendicular transverse of ultraparallel lines

Theorem 1.5 (Bolyai) *Every pair of ultraparallel lines has a unique line, perpendicular to both lines (perpendicular transverse).*

Proof. We use without proof, that the angle, formed by a given line and the intersecting rays, emanating from a point, not on the line, changes continuously. Let the $\overrightarrow{AK}$ and $\overrightarrow{BL}$ be two rays on two ultraparallel lines, such that $BAK\angle = \frac{\pi}{2}$ and $ABL\angle < \frac{\pi}{2}$. Then exists a point M on the ray $\overrightarrow{AK}$ such that $AMB\angle \cong MBL\angle$. Let F be the midpoint of the segment $\overline{BM}$. Let b be a perpendicular line to AK such that F lies on b. Then $TFM\triangle \cong SFB\triangle \Rightarrow BSF\angle = \frac{\pi}{2}$. ■

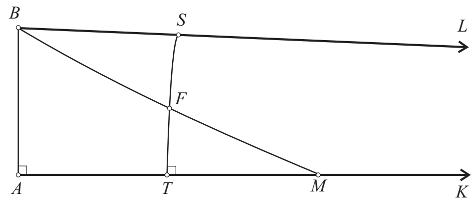


Figure 1.3: Proof: Theorem 1.5

Definition 1.6 *A quadrilateral, with two equal sides perpendicular to the same (base) side called* SACCHERI QUADRILATERAL.

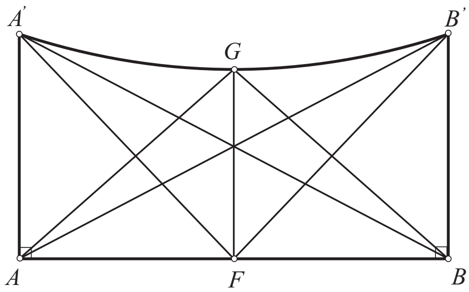


Figure 1.4: Proof: Lemma 1.7

Lemma 1.7 1. $AA'B'\angle \cong BB'A'\angle < \frac{\pi}{2}$

2. $A'G \cong GB' \wedge AF \cong FB \Rightarrow FG \perp AB \wedge FG \perp A'B'$

3. $|\overline{A'B'}| > |\overline{AB}|$

Proof. $BAB'\triangle \cong ABA'\triangle$, because $\overline{AB}$ is common, $\overline{AA'} \cong \overline{BB'}$ and $A'AB\angle \cong B'BA\angle = \frac{\pi}{2} \Rightarrow \overline{AB'} \cong \overline{A'B} \wedge B'AB\angle \cong A'BA\angle \Rightarrow A'AB'\angle \cong B'BA'\angle \Rightarrow AA'B'\triangle \cong BB'A'\triangle \Rightarrow$ *1.*

Similarly, $AA'F\triangle \cong BB'F\triangle$ and therefore $A'FG\triangle \cong B'FG\triangle \Rightarrow A'GF\angle \cong FGB'\angle = \frac{\pi}{2} \Rightarrow$ *2.*

Consider now the triangles $A'AB\triangle$ and $B'BA'\triangle$. Since $B'BA'\angle + A'BA\angle = \frac{\pi}{2}$ and $AA'B\angle + A'BA\angle < \frac{\pi}{2} \Rightarrow AA'B\angle < B'BA\angle$. Using the Arm-lemma, we obtain *3.* ■

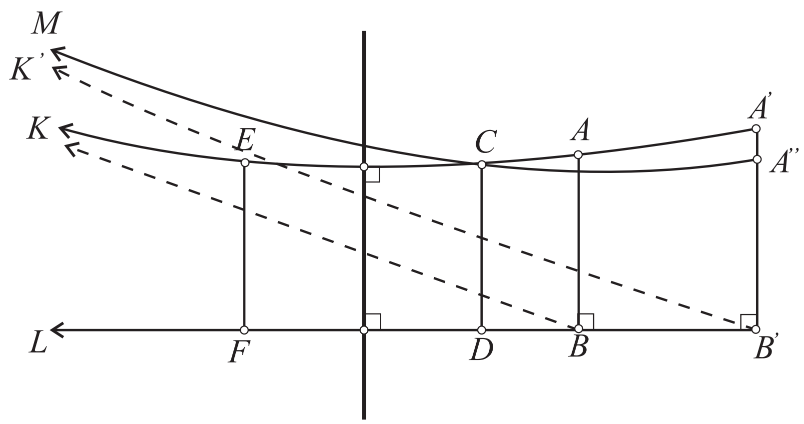


Figure 1.5: Hilbert construction

Construction (Hilbert): Suppose that $AB \perp r$, $KAB\angle < \frac{\pi}{2}$, (KAA'), $A'B' \perp r$ and $|\overline{A''B}| = |\overline{AB}|$. Then $(A'A''B')$, because $ABB'A''$ is a Saccheri quadrilateral $\Rightarrow BAA''\angle < \frac{\pi}{2} < BAA'\angle$. Let s' be a line such that $KAB\angle \cong K'A''B'\angle$. Then $s \cap s' \neq \emptyset$, since $\overrightarrow{BL} \cap \overrightarrow{AK} = \emptyset \Rightarrow \overrightarrow{BK} \parallel \overrightarrow{AK}$ and $\overrightarrow{B'K'} \parallel \overrightarrow{A''K'}$, furthermore $LBK\angle \cong LB'K'\angle$. Then $\overrightarrow{BK}$ and $\overrightarrow{B'K'}$ are ultraparallel rays $\Rightarrow s \cap B'K' \neq \emptyset \Rightarrow P$. Using the Pasch axiom in $BAA'\triangle$ with $B'K'$ we obtain that $B'K'$ intersects either $\overline{AA'}$ or $\overline{AB}$. If it is $\overline{AA'}$, then we are done, otherwise in $KBA\triangle$ we use the Pasch axiom again, but $B'K'$ cannot intersect $BK \Rightarrow$ it must intersect AK. Let C be $s \cap s'$ and $D \in r$ b such that $CD \perp r$. Construct a point E on s so that $\overline{AE} \cong \overline{A''C}$. Then let $F \in r$ be such that $EF \perp r$. Then $EFCD$ is a Sacchery quadrilateral $\Rightarrow$ the symmetry axis is good. ■

1.2 Line pencils and cycles

Definition 1.8 *The lines pencils are set of lines:*

- *passing through a given (finite) point*
- *parallel to a given line*
- *perpendicular to a given line.*

Definition 1.9 *A cycle is the orbit of a point, reflecting it to a given line pencil:*

- *cycle (finite point)*
- *horocycle/paracycle (parallel)*
- *hypercycle/equidistant line (perpendicular).*

Theorem 1.10 *If, a cycle has three collinear points, then it is a line.*

Proof. Let $P'$and P'' be the reflections of the point P respected to two elements of the line pencil $\Rightarrow$ they are the perpendicular bisector of the the segments $\overline{PP'}$ and $\overline{PP''}$. We have the perpendicular transverse of these lines $\Rightarrow$ ultraparallel lines $\Rightarrow$ the points lie on a hypercycle $\Rightarrow$ the point lie on the base line. ■

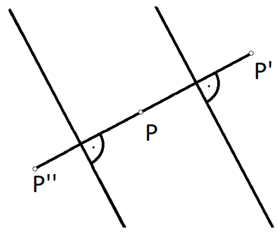


Figure 1.6: Proof: Theorem 1.10

Definition 1.11 *Let A and B be two points on two parallel rays. We say that A and B are corresponding points if the segment $\overline{AB}$ forms equal angles with the rays.*

Lemma 1.12 Circle*: Locus of point, that are equidistant from a given point.*
Paracycle*: Locus of points, that are corresponding to a given point, respected to a given parallel line pencil.*
Hypercycle*: Locus of points, that are equidistant from a given point.*

1.3 Models of the hyperbolic geometry

1. CAYLEY-KLEIN DISK MODEL
   Points: Interior of the unit disk
   Lines: Chords
   Axioms: I-IV trivial, since it is part of the Euclidean plane
   Parallels: Chords, sharing an endpoint (boundary point)
   Perpendiculars: $f \perp g$ if and only if f goes through the intersection point of the tangents of g.

2. POINCARÉ DISK MODEL /conformal disk model/
   Points: Interior of the unit disk
   Lines: Diameters and circular arcs, perpendicular to the model circle
   Axioms: I-IV trivial, since it is part of the Euclidean plane
   Parallels: Circular arcs and diameters, sharing an endpoint (boundary point)

3. POINCARÉ HALF-PLANE MODEL
   Points: Upper half-plane
   Lines: Rays and circular arcs, perpendicular to the model circle
   Axioms: I-IV trivial, since it is part of the Euclidean plane
   Parallels: Circular arcs and rays, sharing an endpoint (boundary point)

Remark 1.13 *Both the Poincaré disk and half-plane model are conformal models: The angle of lines seems real size in these models.*

1.3.1 Orthogonality in the Cayley-Klein model

Lemma 1.14 *If, the length of the tangents, drawn from an external point to two intersecting circles, are equal, then the point lies on the common secant of the circles.*

Proof. Let K be the external point and $|\overline{KT}| = |\overline{KR}|$. The ray, emanating from K through one of the intersection point A intersects the circles in points B and C. Using the intersecting secant theorem: $|\overline{KA}||\overline{KB}| = |\overline{KT}|^2 = |\overline{KR}|^2 = |\overline{KA}||\overline{KC}| \Rightarrow |\overline{KB}| = |\overline{KC}| \Rightarrow B = C$. ■

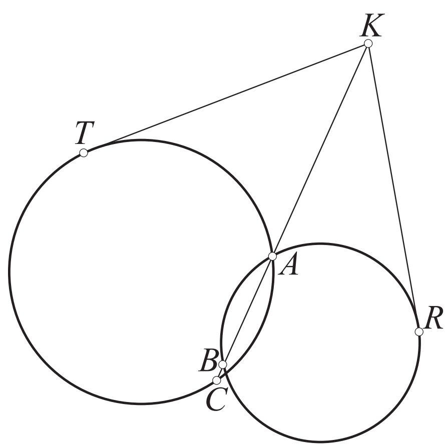


Figure 1.7: Proof: Lemma 1.14

Definition 1.15 *Two chords of a said to be conjugate to each other if the intersection point of the tangent, drawn to the circle at the endpoints of one of the chords lies on the line of the other chord.*

Lemma 1.16 *Two lines are orthogonal to each other in the Cayley-Klein model if and only if the representing chords are conjugate to each other.*

Proof. Let KL and $K'L'$ be orthogonal lines in the Poincaré disk model, intersecting each other at the point P. Then $|\overline{OP}| = |\overline{OK'}| = |\overline{OL'}|$, therefore O must lie on the common secant of the circles, determined by the points $KK'LL'$ and KPL, for the previous lemma. ■

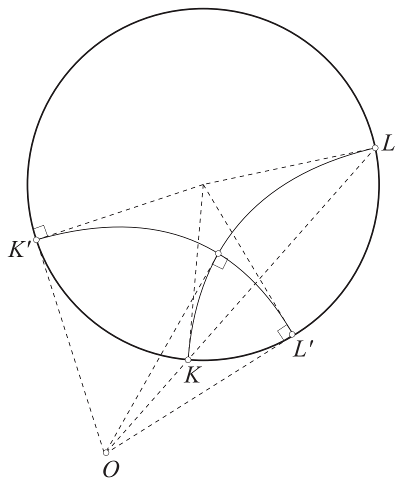


Figure 1.8: Proof: Lemma 1.16

1.3.2 Stereographic projection

1. Defined as the projection of the sphere from the North/South pole onto the equatorial plane.

2. This is a bijective mapping between the unit sphere and the plane (usualy $S^2 \to \mathbb{R}^2/\mathbb{C}$).

3. It is equivalent with the projection from the North pole onto the tangent plane of the South pole.

Theorem 1.17 *The stereographic projection preserves circles and it is conformal.*

Proof. Let P and Q be two points on the surface of the sphere, and their projections are P' and Q'. Let the intersection of the line PQ and $P'Q'$ be N, and S be the intersection of the line PQ and the tangent plane of the center of the projection O. First, we prove the $PQQ'P'$ is a cyclic quadrilateral. OS is the tangent line of the triangle $OQP\triangle$. Because of the inscribed angle theorem, the supplementary angle of $QOS\angle$ is equal to $QPO\angle$. Since $OS \parallel Q'N$, the supplementary angle of $QOS\angle$ is also equal to $OQ'P'\angle$. Therefore $|\overline{NP}||\overline{NQ}| = |\overline{NP'}||\overline{NQ'}|$.
Now let $|\overline{PQ}|$ be a chord of a circle on the sphere. Then the product $|\overline{NP}||\overline{NQ}|$ is constant for every $|\overline{PQ}|$ chord and equal to the product $|\overline{NP'}||\overline{NQ'}|$. This is true for every point N, which lies on the intersection line of the projection plane and the plane, that contains the circle. The intersection of the equatorial plane with the elliptical cone of the circle and O is also a circle.

Conformal: Let P be a vertex of an angle with tangent lines t_1 and t_2. Let k_i be a spherical circle in the plane of t_i and O such that P and O lie on k_i. Then the tangent of these circles s_i meet at the same angle and they lie a parallel plane to the tangent plane of O. Therefore t'_1 and t'_2 also meet at the same angle. ■

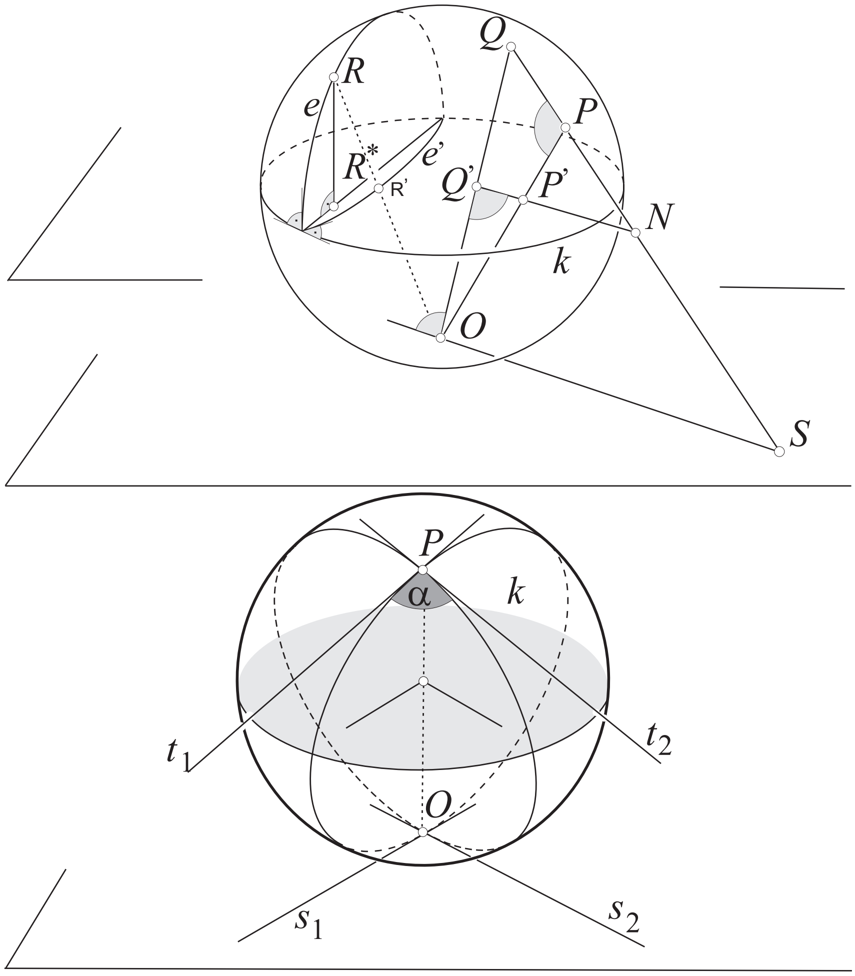


Figure 1.9: Proof: Theorem 1.17

Theorem 1.18 *Consider the unit disk with the Poincaré structure. Then, the composition of the inverse stereographic projection with orthogonal projection back to the plane of the circle is a bijection on the disk, the ideal points are fix points and it results in the Cayley-Klein model.*

Proof. Only the lines are different in the two models. Since a line in the Poincaré disk model is a circle, orthogonal to the great circle, the inverse stereographic projected image is also a circle, orthogonal to the main circle. The plane of it is perpendicular to the base plane ⇒ the orthogonal projection of the circular arc is a line segment, connecting the endpoints of the Poincaré line (circle). ■

1.3.3 Inversion

Definition 1.19 *Let O be the center of a circle/sphere with radius r. The inverse image of a point $P(\neq O)$ is the point P' if P' lies on the ray $\overrightarrow{OP}$ and $|\overline{OP}||\overline{OP'}| = r^2$. The mapping, that assigns the inverse image to every point of the plane/space, is called inversion.*

Theorem 1.20 *The image of a circle or a line by inversion is either a circle or a line. Inversion is a conformal mapping.*

Proof.

- The base circle of the inversin is fixed point by point.
- Every line, through the center of the inversion is also invariant (inside $\longleftrightarrow$ outside).
- The image of a line, which does not contain O, is a circle, through O:
  Let P be the foot point of the orthogonal line to the given line, through O and Q be an arbitrary point on the given line. Let P' and Q' be the inverse image of P and Q respectively. Then $|\overline{OP}||\overline{OP'}| = |\overline{OQ}||\overline{OQ'}| \Rightarrow \frac{|\overline{OP}|}{|\overline{OQ}|} = \frac{|\overline{OQ'}|}{|\overline{OP'}|} \Rightarrow OP'Q'\triangle \sim OQP\triangle \Rightarrow P'Q'O\angle = \frac{\pi}{2} \Rightarrow Q'$ lies on the Thales circle above the segment $\overline{OP'}$ as diameter.
- The image of a circle, which contains O is a line, which does not contain O.
- The image of a circle, which does not contain O, is also a circle, which does not contain O:
  Let PQ be a secant line through O. Then the product of $|\overline{OP}||\overline{OQ}| = p$ is a constant independent from the secant. Applying a scaling from O by ratio $1 : p$, we obtain the l^* circle with center K^*. The secant intersects l^* and the image of Q is Q'. Then $|\overline{OQ^*}||\overline{OP}| = \frac{|\overline{OQ}|}{p}|\overline{OP}| = 1$

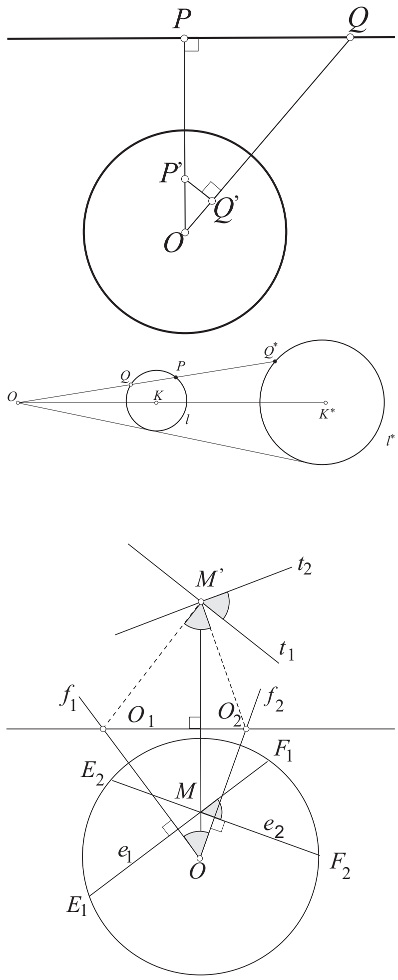


Figure 1.10: Proof: Theorem 1.20

Conformal: Let e_1 and e_2 be the tangents to the curves, their intersection be M and the image of M be M'. Let f_i be perpendicular lines to e_i through O, and O_i be the intersection points of f_i by the perpendicular bisector of $\overline{OM'}$. Then the images of e_i by the inversion are circles with center O_i through O. Then $t_i \perp O_i M'$, therefore $t_1 t_2\angle \cong O_1 M' O_2\angle \cong O_1 O O_2\angle \cong e_1 e_2\angle$. ■

Remark 1.21 *Applying the inversion to the Poincaré disk/sphere model at a boundary point, we obtain the Poincaré half-plane/half-space model. We also have the advantage, that this point is arbitrary, therefore we can choose the representative of one line/plane in the model to be a ray/half plane. Because the Poincaré disk/sphere model is a conformal model and inversion is a conformal mapping, the Poincaré half-plane/half-space model is also a conformal model.*

1.4 Distance and angle of space elements

1.4.1 Mutual position of space elements ($\mathbf{E}^3$)

The parallelism of lines is an equivalence relation (parallel line pencil). We assign an ideal point for every equivalence class. Two line intersect each other at this ideal point, if they are parallel in the Euclidean space. The union of ideal points forms an ideal plane.

Definition 1.22 *The projective space* $\mathbf{P}\mathbb{R}^3$ *is the union of the Euclidean space and the ideal elements.*

1. Line-Line
   - intersecting: if their intersection is not ideal
   - parallel: if they are not intersecting in $\mathbf{E}^3$ but in $\mathbf{P}\mathbb{R}^3$
   - skew: if they are not intersecting in $\mathbf{P}\mathbb{R}^3$
2. Line-Plane/Plane-Plane
   - intersecting
   - parallel

1.4.2 Mutual position of space elements ($\mathbf{H}^3$)

1. Line-Line
   - intersecting
   - parallel
   - ultraparallel
   - skew (not intersecting in $\mathbf{P}\mathbb{R}^3$)
2. Line-Plane
   - intersecting
   - parallel
   - skew (they intersect each other either outside of the model or only in $\mathbf{P}\mathbb{R}^3$)

3. Plane-Plane
    - intersecting
    - parallel
    - ultraparallel

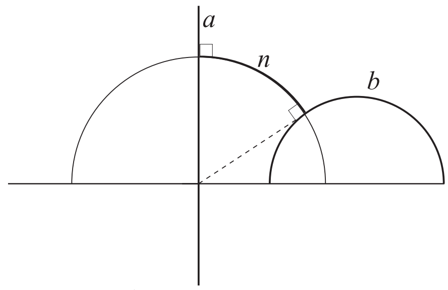
1.4.3 Perpendicular transverse of non-intersecting hyperbolic space elements

Theorem 1.23 *In the hyperbolic space, there exits a unique perpendicular transverse line for two skew lines, for two ultraparallel lines/planes.*

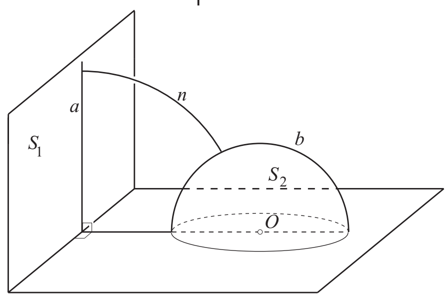
Proof. We use the Poincaré half-plane/half-space model.

1. *Ultraparallel lines (planar case)*

2. *Ultraparallel line-plane*
   Perpendicular line in the boundary plane to the line through the center of the plane (sphere).



3. *Ultraparallel planes*
   We draw a perpendicular line to the intersection of the plane and the boundary plane through the center of the sphere. We apply the previous case for the sphere and the line, perpendicular to the boundary plane in the given plane through the footpoint of the previous line.



4. *Skew lines*
   Let k be a half-sphere, perpendicular to the boundary plane, through $A, B, C \Rightarrow b \in k$. FD is the perpendicular bisector of $\overline{BC} \Rightarrow S = AD \cap BC$. Let N be a point on b uch that $N^* = S$. Then n will be a circle perpendicular both a and b with center A and radius $|\overline{AN}|$. The perpendicularity to a is obvious. Let t_b and t_n be the tangents of b and n in N. t_b is in the plane of $BNC \Rightarrow t_b \perp FD, NF \Rightarrow t_b \perp ND$. In the triangle $AND\triangle$ AD is the diameter of the circumscribed circle $\Rightarrow AND\angle = \frac{\pi}{2} \Rightarrow t_n = ND \Rightarrow t_n \perp t_b \Rightarrow n \perp b$. ■

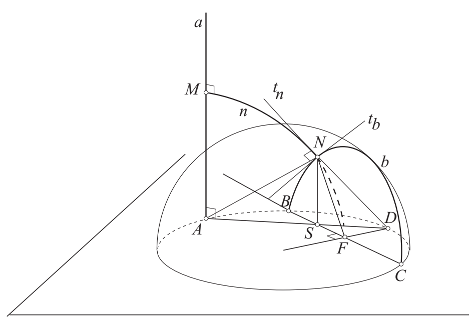


Figure 1.11: Proof: Theorem 1.23

1.4.4 Cross-ratio

Definition 1.24 *Let z_1, z_2, z_3, z_4 be complex numbers. Then the cross-ratio:*

$$(z_1, z_2, z_3, z_4) := \frac{z_3 - z_1}{z_2 - z_3} : \frac{z_4 - z_1}{z_2 - z_4}.$$

Theorem 1.25 *The cross-ratio of 4 distinct points is real, if and only if they all lie either on a circle or on a line.*

Proof. We use the polar decomposition: $z = re^{i\Theta}$

$$\frac{z_3 - z_1}{z_2 - z_3} : \frac{z_4 - z_1}{z_2 - z_4} = \frac{z_1 - z_3}{z_2 - z_3} \cdot \frac{z_4 - z_2}{z_4 - z_1} =$$

$$= \left|\frac{z_1 - z_3}{z_2 - z_3}\right| e^{i\Theta_3} \left|\frac{z_1 - z_4}{z_2 - z_4}\right| e^{i\Theta_4} = Re^{i(\Theta_3+\Theta_4)}$$

where R is a real number. $e^{i(\Theta_3+\Theta_4)} \in \mathbb{R} \Leftrightarrow \Theta_3 + \Theta_4 = k\pi (k \in \mathbb{Z})$. If $\Theta_3 + \Theta_4 = 2k\pi$ then they lie on a circle, otherwise they lie on a line. ■

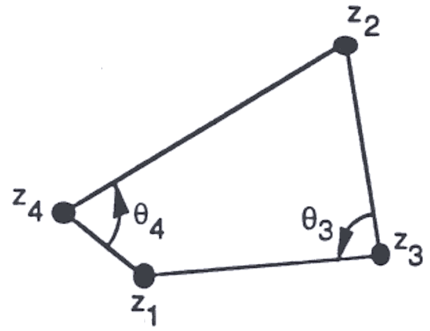


Figure 1.12: Proof: Theorem 1.25

1.4.5 Distances in the hyperbolic space

Definition 1.26 *The hyperbolic distance in the Poincaré disk model: $d(X,Y) = \log(X,Y,U,V)$, where U and V are the endpoints of the line, determined by X and Y such that X, Y, U, V is the cyclic order of the points on the representing circle.*

– point-point: definition
– point-line: distance of the given point and the footpoint of the orthogonal line to the given line through the given point
– point-plane: distance of the given point and the footpoint of the orthogonal line to the given plane through the given point
– line-line: length of the perpendicular transverse line segment
– line-plane: length of the perpendicular transverse line segment
– plane-plane: length of the perpendicular transverse line segment

1.4.6 Angle and distance in the Cayley-Klein model

– Angle: We follow the mapping from the Poincaré structure to the Cayley-Klein structure:

$\alpha = (u,v)\angle = (u',v')\angle = VP'U\angle$

$|\overline{UV}|^2 = |\overline{P'U}|^2 + |\overline{P'V}|^2 - 2|\overline{P'U}||\overline{P'V}|\cos(\alpha)$

$|\overline{P'U}|^2 = r_1^2 = u_1^2 + u_2^2 - 1,\ |\overline{P'V}|^2 = r_2^2 = v_1^2 + v_2^2 - 1$

$|\overline{UV}|^2 = (u_1 - v_1)^2 + (u_2 - v2)^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2u_1v_1 - 2u_2v_2$

$$\frac{r_1^2 + r_2^2 - |\overline{UV}|^2}{2r_1r_2} = \frac{-2 + 2u_1v_1 + 2u_2v_2}{2\sqrt{(-1 + u_1^2 + u_2^2)(-1 + v_1^2 + v_2^2)}} =$$

$$\cos(\alpha) = \frac{-1 + u_1v_1 + u_2v_2}{\sqrt{(-1 + u_1^2 + u_2^2)(-1 + v_1^2 + v_2^2)}} \qquad \text{(C-K}\angle)$$

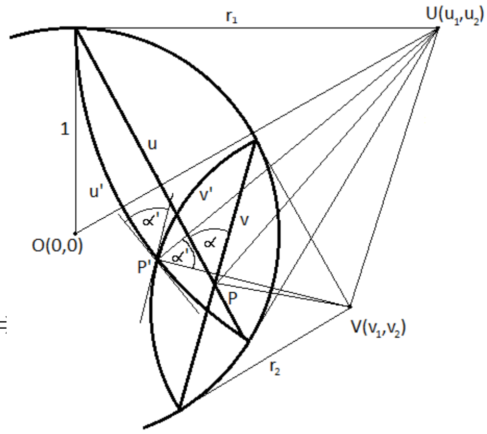


Figure 1.13: Angle in the Cayley-Klein model

– Distance:

$$\cosh(d(\mathbf{x},\mathbf{y})) = \frac{-1 + x_1y_1 + x_2y_2}{\sqrt{(-1 + x_1^2 + x_2^2)(-1 + y_1^2 + y_2^2)}}$$

1.4.7 Hyperboloid model

We use the $\mathbf{V^3}$ real vector space with the standard $\{e_1, e_2, e_3\}$ base. We introduce the symmetric bilinear form: $\langle e_1, e_1\rangle = 1,\ \langle e_2, e_2\rangle = 1,\ \langle e_3, e_3\rangle = -1,\ \langle e_i, e_j\rangle = 0,\ (i \neq j)$. If $\mathbf{x}, \mathbf{y} \in \mathbf{V}^3$, then $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + x_2y_2 - x_3y_3$, and

$$\langle \mathbf{x}, \mathbf{x}\rangle \begin{cases} < 0 & \text{time} - \text{like} \\ = 0 & \text{light} - \text{like} \quad \Rightarrow \\ > 0 & \text{space} - \text{like} \end{cases}$$

$$\langle \mathbf{x}, \mathbf{x}\rangle = r^2 \begin{cases} \text{two} - \text{sheeted hyperboloid} & r \in \mathbb{C} \\ \text{cone} & r = 0 \\ \text{one} - \text{sheeted hyperboloid} & r \in \mathbb{R} \end{cases}$$

We define an equivalence relation: $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \exists c \in \mathbb{R} \setminus \{0\} : \mathbf{y} = c\mathbf{x} \Rightarrow$ representing elements: $\mathbf{x}' \sim (x_1, x_2, 1)$.

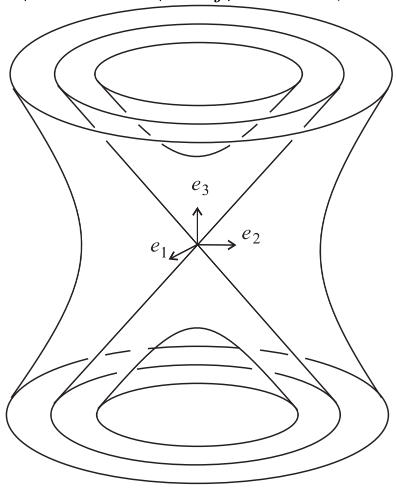


Figure 1.14: Hyperboloids

We assign the time-like vectors to the points of the Cayley-Klein model of the hyperbolic geometry by this equivalence relation.

We define the distance of two points by the $d(X,Y) = \frac{1}{2}\log(X,Y,U_1,U_2)$, where U_1 and U_2 are the boundary points on the line of X and Y such that X, Y, U_1 and U_2 are in cyclic order. Then

$$e^{2d(X,Y)} = (X,Y,U_1,U_2) = \frac{\beta_1}{\alpha_1} : \frac{\beta_2}{\alpha_2} \text{ where } \mathbf{u_1} = \alpha_1\mathbf{x} + \beta_1\mathbf{y},\ \text{and } \mathbf{u_2} = \alpha_2\mathbf{x} + \beta_2\mathbf{y}.$$

$$\langle u_i, u_i \rangle = \alpha_i^2 \langle \mathbf{x}, \mathbf{x} \rangle + 2\alpha_i \beta_i \langle \mathbf{x}, \mathbf{x} \rangle + \beta_i^2 \langle \mathbf{y}, \mathbf{y} \rangle = 0 \Rightarrow$$
$$\langle \mathbf{x}, \mathbf{x} \rangle + 2\tfrac{\beta_i}{\alpha_i} \langle \mathbf{x}, \mathbf{x} \rangle + \left(\tfrac{\beta_i}{\alpha_i}\right)^2 \langle \mathbf{y}, \mathbf{y} \rangle = 0$$

$$\left(\frac{\beta_i}{\alpha_i}\right)_{1,2} = \frac{-2 \langle \mathbf{x}, \mathbf{y} \rangle \pm \sqrt{4 \langle \mathbf{x}, \mathbf{y} \rangle^2 - 4 \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}{2 \langle \mathbf{y}, \mathbf{y} \rangle} \Rightarrow$$

$$e^{2d} = (\mathbf{x}, \mathbf{y}, \mathbf{u}_1, \mathbf{u}_2) = \underbrace{\frac{\langle \mathbf{x}, \mathbf{y} \rangle - \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}{\langle \mathbf{x}, \mathbf{y} \rangle + \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}}_{a}$$

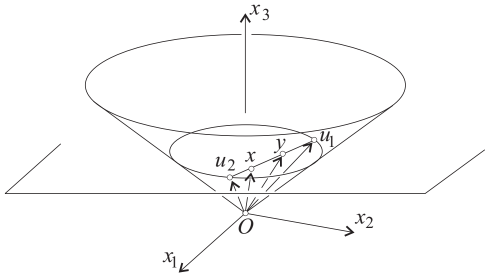


Figure 1.15: Distance in the hyperboloid model

$$\cosh(d(\mathbf{x}, \mathbf{y})) = \frac{e^d + e^{-d}}{2} = \frac{\sqrt{a} + \frac{1}{\sqrt{a}}}{2}, \text{ but } \sqrt{a} + \frac{1}{\sqrt{a}} = \sqrt{(\sqrt{a} + \frac{1}{\sqrt{a}})^2} = \sqrt{a + \frac{1}{a} + 2}$$

$$a + \frac{1}{a} + 2 = \frac{\langle \mathbf{x}, \mathbf{y} \rangle - \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}{\langle \mathbf{x}, \mathbf{y} \rangle + \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}} + \frac{\langle \mathbf{x}, \mathbf{y} \rangle + \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}{\langle \mathbf{x}, \mathbf{y} \rangle - \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}} + 2 =$$

$$= \frac{\left(\langle \mathbf{x}, \mathbf{y} \rangle - \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}\right)^2 + \left(\langle \mathbf{x}, \mathbf{y} \rangle + \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}\right)^2}{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \left(\sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}\right)^2} + 2 =$$

$$= \frac{2 \langle \mathbf{x}, \mathbf{y} \rangle^2 + 2 \left(\sqrt{\langle \mathbf{x}, \mathbf{y} \rangle^2 - \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}\right)^2}{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle} + 2 = \frac{4 \langle \mathbf{x}, \mathbf{y} \rangle^2 - 2 \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle} + 2 =$$

$$= \frac{4 \langle \mathbf{x}, \mathbf{y} \rangle^2}{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle} \Rightarrow \cosh\left(\frac{1}{2} \log(X, Y, U_1, U_2)\right) = \cosh(d(X, Y)) = \frac{-\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}.$$

1.4.8 Pole-polar relation

To represent a line u in the hyperboloid model, we consider a plane, orthogonal to $\mathbf{u}(u_1, u_2, 1)$ through the origin: $xu_1 + yu_2 + z = 0$. Intersecting it with the plane $z = 1$, we get a line with the equation $xu_1 + yu_2 + 1 = 0$. If $u_1^2 + u_2^2 > 1$, then it can be assigned as a proper line in the Cayley-Klein model, and $X(\mathbf{x})$ lies on it, if $xu_1 + yu_2 + 1 = 0$. The pole $V(v_1, v_2, 1)$ of this line u is the intersection of the tangents to the boundary point of u. This is also the center of the circle, with determines u in the Poincaré model. To get the radius r of this circle, we apply the Pythagorean theorem: $r^2 = a^2 - 1 = v_1^2 + v_2^2 - 1$.

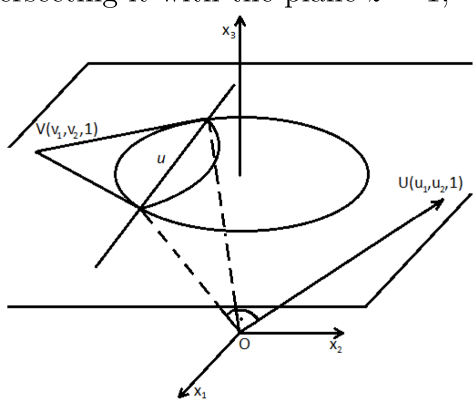


Figure 1.16: Pole-polar

Lemma 1.27 *A line* $\mathbf{u}(u_1, u_2, 1)$ *has the pole* $(-u_1, -u_2, 1)$ *and the point* $P(x, y, 1)$ *lies on it, if and only if* $\mathbf{p} \cdot \mathbf{u} = 0$.

Proof. The equation of the two circles are the following:

$$(x - v_1)^2 + (y - v_2)^2 = v_1^2 + v_2^2 - 1$$
$$x^2 + y^2 = 1$$

Expanding the first equation, we can simplify it:

$$x^2 - 2xv_1 + \cancel{v_1^2} + y^2 - 2yv_2 + \cancel{v_2^2} = \cancel{v_1^2} + \cancel{v_2^2} - 1$$
$$x^2 + y^2 = 1$$

Finally, we get the equation of the radical line of the circles: $-2xv_1 - 2yv_2 = 2 \Rightarrow -xv_1 - yv_2 = 1$. But the radical line is our original u line with the equation $xu_1 + yu_2 + 1 = 0$, therefore $v_1 = -u_1$ and $v_2 = -u_2$. ∎

Remark 1.28 *According to (C-K∠), the angle of two lines* $\mathbf{u}(u_1, u_2, 1)$ *and* $\mathbf{v}(v_1, v_2, 1)$ *can be computed by their poles* $(-u_1, -u_2, 1)$ *and* $(-v_1, -v_2, 1)$*. With the defined bilinear form:*

$$\frac{-1 + (-u_1)(-v_1) + (-u_2)(-v_2)}{\sqrt{(-1 + (-u_1)^2 + (-u_2)^2)(-1 + (-v_1)^2 + (-v_2)^2)}} = \frac{-1 + u_1v_1 + u_2v_2}{\sqrt{(-1 + u_1^2 + u_2^2)(-1 + v_1^2 + v_2^2)}} \Rightarrow$$

$$\cos(\alpha) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\sqrt{\langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle}}.$$

1.5 Models of the Euclidean plane

– Horosphere: We consider the rays, orthogonal to the base plane in the Poincaré half-space model. This is a parallel line pencil. The corresponding cycles are horospheres, represented as planes, parallel to the base plane. Because this is a conformal model, the interior angel sum of any triangle on this plane is π. The geometry on this horosphere is Euclidean.

Theorem 1.29 *Any two horosphere are congruent to each other.*

– Hypercycle: Let O be a point on the Euclidean plane and K be a center of a unit ball, tangent to the plane at O. We consider a unit disk, such that its plane is parallel to the base plane, through K, as the Cayley-Klein model of the hyperbolic plane. The points of the model are the points of the unit disk, and the lines are the lines, through K and the corresponding hypercycles, represented as ellipses. We project this elliptic arc orthogonally onto the sphere (we get a great arc). Then we project it back to the base plane through K.

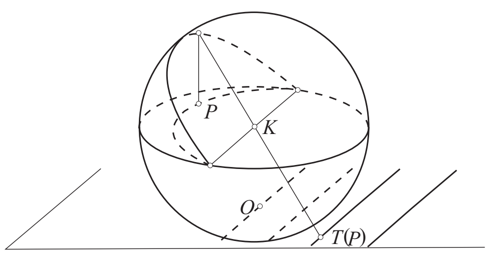


Figure 1.17: Hypercycle model

With the $d(A, B) := |\overline{T(A)T(B)}|$ metric, the congruence axioms will be true, and the Euclidean axiom of parallelism is obviously ture.

1.6 Area on the hyperbolic plane

Definition 1.30 *A triangle is called asymptotic/doubly asymptotic/triply asymptotic, if one/two/three vertex/vertices is/are boundary point(s).*

Theorem 1.31 *All the triply asymptotic triangles are congruent to each other.*

Definition 1.32 *Area is an isometry-invariante, non-negative, additive set function for simply polygons and $T(=\pi)$ is assigned to the triply asymptotic triangle.*

Theorem 1.33 *Any asymptotic triangle can be cur off into a pentagon.*

Proof. (Poincaré disk-model): Let $ABC\triangle$ be such that C is the ideal point and A is the center of the disk. Let D be an ideal point such that (ABD), and M be the footpoint of the perpendicular line to CD through A. Let A_1 be the reflection of B in the line AM and the intersection of BC and A_1D be M_1. Let the footpoints of the perpendicular line to DC through B and A_1 be Q and P respectively. If the reflection of BC in A_1P is DA_2, then the triangle $M_2A_1A_2\triangle/4/$ is congruent to $A_1M_1M_2\triangle/3/$ and $M_1M_2B\triangle$. Continuing this process, we get $ABQPA_1$ pentagon. ■

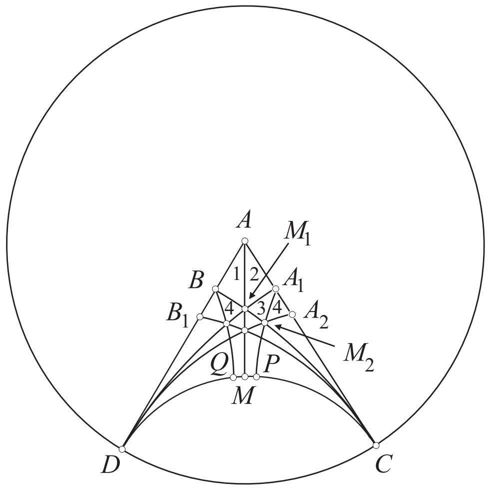


Figure 1.18: Lindberg method

Theorem 1.34 *If, the vertex angle of a doubly asymptotic triangle is α, then the area of it is $c(\pi-\alpha)$ $/c \in \mathbb{R}^+/$.*

Proof. Let $f(\phi)$ be the area, if $\phi = \pi - \alpha$ is the supplementary angle. The union of the two corresponding doubly asymptotic triangle is a triply asymptotic triangle (see Figure 1.19), therefore: $f(\phi) + f(\pi - \phi) = T$.

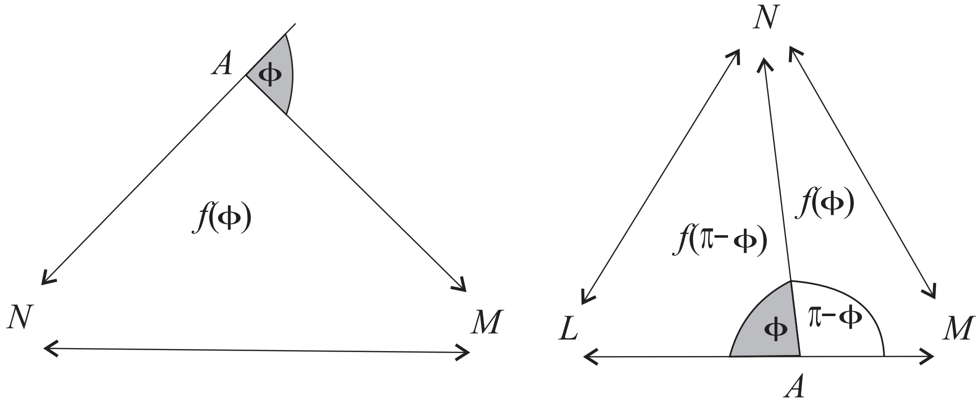


Figure 1.19: $f(\phi)$ is additive

Figure 1.20 shows us, that $T = f(\phi) + f(\varphi) + f(\pi - \phi - \varphi) \Rightarrow f(\phi) + f(\varphi) = f(\phi + \varphi)$

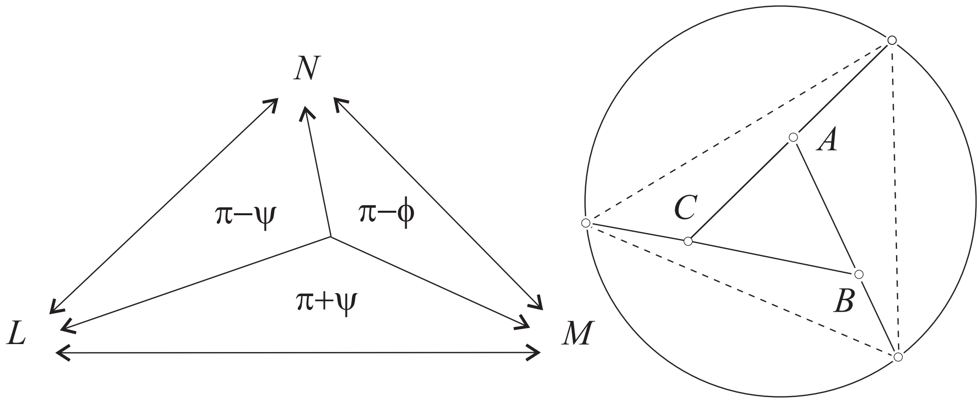


Figure 1.20: Area and defect

Our only solution is $f(x) = \lambda x$, since f is monotonously increasing and $f(1) = \lambda \Rightarrow f(n) = n\lambda$. Now, if $\frac{k}{n} \leq x \leq \frac{k+1}{n} \Rightarrow k \leq nx \leq k+1 \Rightarrow k\lambda \leq nf(x) \leq (k+1)\lambda \Rightarrow \frac{k}{n} \leq \frac{f(x)}{\lambda} \leq \frac{k+1}{n} \Rightarrow \left|x - \frac{f(x)}{\lambda}\right| \leq \frac{1}{n} \forall n \Rightarrow f(x) = \lambda x$. ■

Remark 1.35 *Because the triply asymptotic triangle is $T \Rightarrow \lambda = \frac{T}{\pi}$. Therefore, we choose the value of T be π.*

Theorem 1.36 *The area of any hyperbolic triangle is its defect.*

Proof. We make up the $ABC\triangle$ to a triply asymptotic triangle by expanding the sides cyclically (see Figure 1.20). The area of the three extra doubly asymptotic triangles are α, β and γ respectively. Therefore, the area can be expressed from the formula:

$$\pi = A(ABC) + \alpha + \beta + \gamma \Rightarrow A(ABC) = \pi - (\alpha + \beta + \gamma) = \delta. \quad ■$$