Classification of isometries

1.1 Isometries of the Euclidean plane

Definition 1.1 *Isometry is a distance-preserving bijective mapping of the space to itself.*

Claim 1.2 *The isometries of the space form a non-commutative group.*

Theorem 1.3 *Every isometry is a collineation.*

Proof. Let A, B and C be three collinear points, and their image be A', B' and C'. Assume, that $A'B'C'$ form a triangle. But $AC \cong A'C'$, $BC \cong B'C'$ and $AB \cong A'B' \Rightarrow |\overline{AC}| + |\overline{BC}| = |\overline{AB}| = |\overline{A'B'}| < |\overline{A'C'}| + |\overline{B'C'}|$. $\Rightarrow\!\Leftarrow$ ■

Theorem 1.4 *If, an isometry has two fix points, then the determined line is also fixed point-by-point (axis).*

Proof. Let $A = A'$ and $B = B'$ be true and P be a point on this line. Then P' must be also on the line and $AP \cong AP' \wedge BP \cong BP' \Rightarrow P = P'$. ■

Theorem 1.5 *If, an isometry has three fix points, then the determined plane is also fixed point-by-point (plane).*

Definition 1.6 *The reflection of the plane in a line t assign the P' point to P if $PP' \perp t$ and $PT \cong P'T$, where $T = PP' \cap t$.*

Theorem 1.7 *Reflection is an isometry.*

Proof. Let C and D be the footpoints of the perpendiculars to t from A and B respectively. Then $BCD\triangle \cong B'CD\triangle \Rightarrow BC \cong B'C \wedge BCD\angle B'CD\angle \Rightarrow ACB\angle \cong A'CB'\angle \Rightarrow ABC\triangle \cong A'B'C\triangle \Rightarrow AB \cong A'B'$ ■

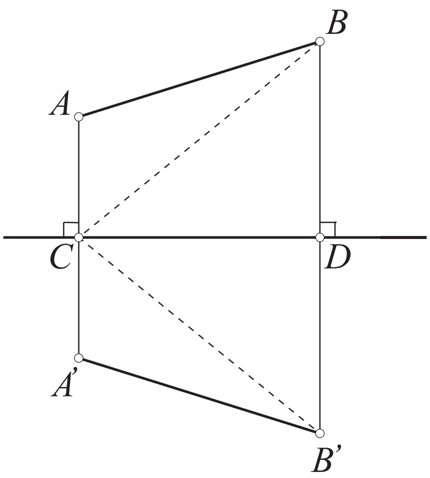

Figure 1.1: Hypercycle model

Claim 1.8 *The composition of a reflection to itself is the identical mapping.*

Theorem 1.9 *If, a non-identical isometry has exactly two fix points, then it is a reflection in the determined line.*

Proof. Let A and B be the fix points and P be an arbitrary point. Then $AP \cong AP' \wedge BP \cong BP' \Rightarrow P = P'$, or AB is the perpendicular bisector of $PP' \Rightarrow d(P, AB) = d(P', AB)$ and $PP' \perp AB$. ■

Theorem 1.10 *If, a non-identical R isometry has exactly one fix point, then it is the composition of two reflections, through this fix point.*

Proof. Let P be an arbitrary point and A be the fix point. Then $AP \cong AP'$, therefore A lies on the perpendicular bisector (t_1) of PP'. The composition of the original isometry and the reflection in this line has two fix points, therefore it is a reflection in a line t_2. Then $R \circ t_1 = t_2 \Rightarrow R = t_2 \circ t_1^{-1} = t_2 \circ t_1$. ■

Theorem 1.11 *If, a non-identical isometry T has no fix point, then it is the composition of either two or three reflections.*

Proof. Let P be an arbitrary point and P' be its image. Then $T \circ t_1$ has either one, or two fix points, where t_1 is the reflection to the perpendicular bisector of PP'.

1. $T \circ t_1 = R = t_2 \circ t_3 \Rightarrow T = t_2 \circ t_3 \circ t_1^{-1} = t_2 \circ t_3 \circ t_1$
2. $T \circ t_1 = t_2 \Rightarrow T = t_2 \circ t_1^{-1} = t_2 \circ t_1$ ■

Theorem 1.12 *Any isometry of the plane is the composition of at most three reflections in a line.*

Proof. It follows from the above theorems trivially, since any isometry has either 0, 1, or 2 fix points or all the points on the plane is fixed point-by-point. ■

Lemma 1.13 *The composition of two reflections can be considered as either a rotation (intersecting axis) or a translation (parallel axis).*

Proof.
– Intersecting axis: Let A be the intersection of the two lines t_1 and t_2. Then $PAt_1\angle \cong P'At_1\angle$ and $P'At_2\angle \cong P''At_2\angle$, therefore $PAP''\angle = 2t_1t_2\angle = 2\alpha \Rightarrow$ Rotation around A by 2α.
– Parallel axis: Now, we can say that $d(P, t_1) = d(P', t_1)$ and $d(P', t_2) = d(P'', t_2)$ and P, P', P'' are collinear points, therefore $d(P, P'') = d(t_1, t_2) = 2d \Rightarrow$ Translation, perpendicular to t_i with $2d$. ■

Remark 1.14 *In the case of the rotation, only the angle of the lines matters, in the case of the translation, only the distance and the direction matters.*

Theorem 1.15 *Every isometry of the Euclidean plane belongs to one of the following 5 groups:*

1. *Identity (0)*
2. *Reflection (1)*
3. *Rotation (2a)*
4. *Translation (2b)*
5. *Glide reflection (3)*

Proof. We have a full discussion for 2 reflections.

1. case $t_1 \cap t_2 \neq \emptyset$

   (a) $t_1 \cap t_2 = t_2 \cap t_3$
   Then we can consider the $t_2 \circ t_3$ as a rotation around this point, and we can rotate the lines around this center to get $t_1 = t_2'$. Then $t_1 \circ t_2 \circ t_3 = t_1 \circ t_2' \circ t_3' = t_1 \circ t_1^{-1} \circ t_3' = t_3'$.

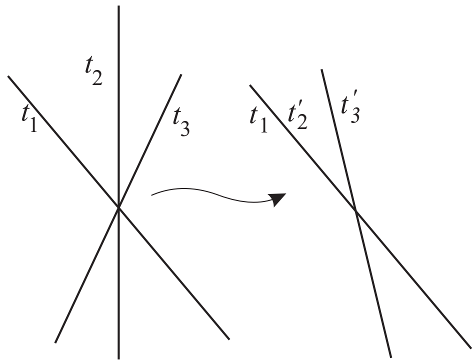

Figure 1.2: Case 1. (a)

   (b) $t_1 \cap t_2 \neq t_2 \cap t_3$
   First, we rotate t_1 and t_2 around their intersection O until $t_2' \perp t_3$. Then we rotate t_2' and t_3 around their intersection K until $t_2'' \parallel t_1'$. Then $t_1' \circ t_2''$ is a translation and $t_1' \circ t_2'' \circ t_3'$ is a glide reflection.

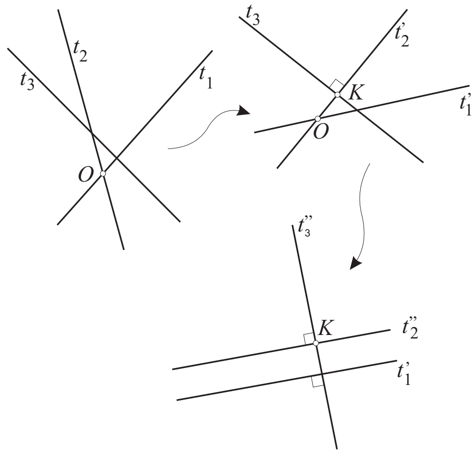

Figure 1.3: Case 1. (b)

2. case $t_1 \cap t_2 = \emptyset \Rightarrow t_1 \parallel t_2$

   (a) $t_1 \parallel t_2 \parallel t_3$
   Then we can translate t_2 and t_3 along their perpendicular line until $t_1 = t_2'$ then their composition will be the identity, therefore only t_3 remains, so this is a translation.

   (b) $t_1 \parallel t_2 \nparallel t_3$
   First, we rotate t_2 and t_3 around their intersection until $t_2' \perp t_1$. Then we rotate t_1 and t_2' around their intersection until $t_1' \parallel t_3'$. Finally, we rotate t_2'' and t_3' around their intersection, until $t_1' \parallel t_2'''$. Then $t_1' \circ t_2''' \circ t_3''$ is a glide reflection. ∎

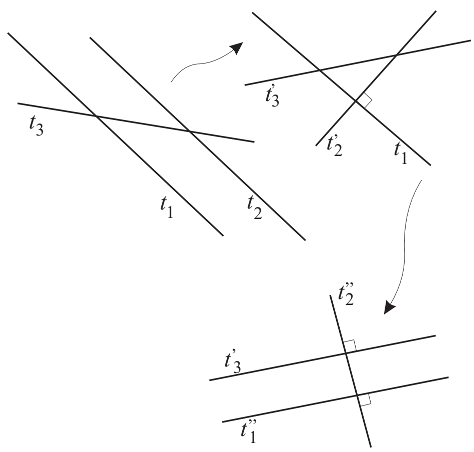

Figure 1.4: Case 2. (b)

Remark 1.16 *Two reflections are commutative to each other if their axis are orthogonal to each other.*

1.2 Isometries of the Euclidean space

Theorem 1.17 *Any isometry of the space is the composition of at most four reflections in a plane.*

Theorem 1.18 *Every isometry of the Euclidean space belongs to one of the following 7 groups:*

1. *Identity (0)*
2. *Reflection (1)*
3. *Rotation (2a)*
4. *Translation (2b)*
5. *Glide reflection (3a)*
6. *Improper rotation (3b)*
7. *Screw displacement (4)*

where

5. *Glide reflection: composition of a translation and a reflection*
6. *Improper rotation: composition of a rotation and a reflection*
7. *Screw displacement: composition of a rotation and a translation*

Definition 1.19 *The reflection in a line in the space means a rotation around this line by* 180°. *This can be represented as a reflection in two orthogonal planes through the line.*

Lemma 1.20 *The composition of two reflections in a line is either rotation or translation or screw displacement, if the two lines are intersecting, parallel or skew lines.*

Proof.
Assume that $e = t_1 \circ t_2$, $f = t_3 \circ t_4$.
1. case: The lines e and f determine a plan
We choose the positions of t_i such that, t_2 and t_3 coincide the determined plane and t_1 and t_4 are orthogonal to it. Then $t_2 \circ t_3$ is identity and if $e \parallel f$, then $t_1 \parallel t_4$, otherwise $t_1 \nparallel t_4$.
2. case: The lines e and f are skew lines
Let n be the perpendicular transverse of e and f. $t_1 := (e, n) \Rightarrow t_2 \perp n$, $t_3 := (f, n) \Rightarrow t_4 \perp n$. Since $t_2 \parallel t_4 \wedge t_3 \perp t_4 \Rightarrow t_3 \perp t_2 \Rightarrow t_1 \circ t_2 \circ t_3 \circ t_4 = (t_1 \circ t_3) \circ (t_2 \circ t_4)$. But n lies both on t_1 and t_3 therefore they determine a rotation, and $t_2 \parallel t_4$ then they determine a translation. The composition of these isometries is a skew displacement. ∎

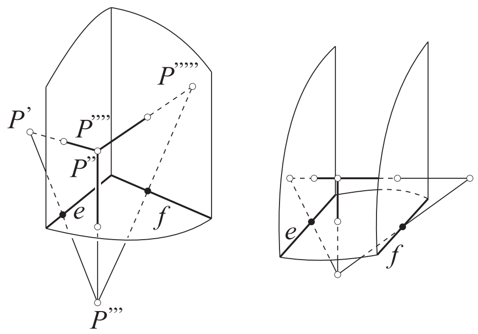

Figure 1.5: 1. case

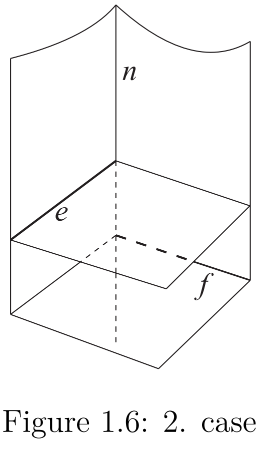

Figure 1.6: 2. case

Lemma 1.21 *The composition of two translations is also translation.*

Proof. Let $t_1 \parallel t_2$ and $t_3 \parallel t_4$ be two translation. If, $t_2 \parallel t_3$, then it is trivial. Otherwise $t_2 \cap t_3 \neq \emptyset \Rightarrow t_2 \cap t_3 = e$. If, we rotate t_2 and t_3 around e by 90°, then $t_2' \perp t_1$ and $t_3' \perp t_4$. Then $= t_1, t_2'$ and t_3', t_4 determines two reflections in a line with parallel axis, therefore it is a translation. ■

Lemma 1.22 *The composition of two rotations with orthogonal axis is a screw displacement.*

Proof. Let t' and t'' be the axis of the rotations and n be their perpendicular transverse line. Then $t' \perp (t'', n)$ and $t'' \perp (t', n)$. Let t_1 and t_2 be reflections such that $t_1 \cap t_2 = t'$ and $t_2 = (t', n)$, furthermore t_3 and t_4 be reflections such that $t_3 \cap t_4 = t''$ and $t_3 = (t'', n)$. Since $t_2 \perp t_3 \Rightarrow t_1 \circ t_2 \circ t_3 \circ t_4 = (t_1 \circ t_3) \circ (t_2 \circ t_4)$. Let e and f be the intersections of t_1, t_3 and t_2, t_4 respectively. But $t_1 \perp t_3$ and $t_2 \perp t_4$, therefore this isometry is can be represented as the composition of two reflections in a line, which axis are in skew position $\Rightarrow$ screw displacement. ■

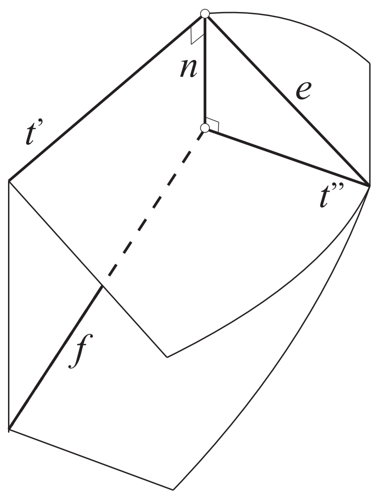

Figure 1.7: Proof of Lemma 1.22

Proof of Theorem 1.18 We have a full discussion for 2 reflections. Composition of at least three reflections:

1. case: The three plane have a common perpendicular plane:
   The orbit of any point lies on the plane, parallel to this orthogonal plane $\Rightarrow$ using the planar case, it is the composition of 3 reflections in line $\Rightarrow$ either reflection or glide reflection $\Rightarrow$ the spatial transformation is also either reflection or glide reflection.

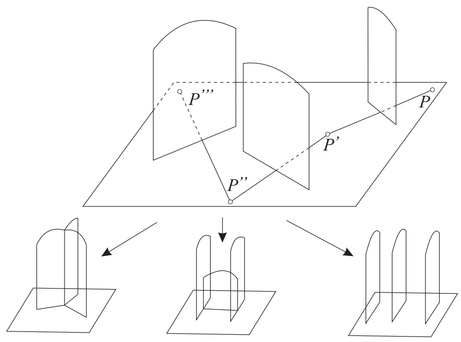

Figure 1.8: 1. case

2. case: If, plane 1 and plane 2 intersect each other in the line m and plane 3 does not intersect m, then the plane orthogonal to m will be orthogonal to plane 3 as well $\Rightarrow$ 1. case. We may assume that plane 3 intersects m in the point M. First we rotate plane 1 and 2 around m such that $2' \perp 3$. Then we rotate $2'$ and $3'$ around their intersection such that $1' \perp 3'$. Finally we get an improper rotation.

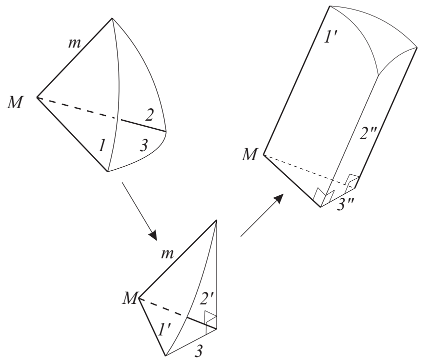

Figure 1.9: 2. case

3. case: If, plane 1 and 2 are parallel to each other, then plane 3 is either parallel them both ⇒ reflection, or plane 3 intersects them in parallel lines. Then the plane, orthogonal to the lines will be orthogonal to all the planes ⇒ 1. case

4. We can assume, that the composition of the first 3 reflections is either glide reflection or improper rotation.
   If, plane 4 is parallel to the third plane of an improper rotation, then it is a screw displacement by definition.
   If, m is the intersection of plane 3 and 4, then this isometry is the composition of two rotation and by lemma, it is a screw displacement.
   Now, we can assume, that the composition of the first 3 reflections is a glide reflection.
   If, plane 4 is parallel to plane 3, then this is the composition of two translations and by lemma, this is another translation.
   If, plane 4 intersects plane 3, then we rotate plane 2 and 3 around their intersection line such that

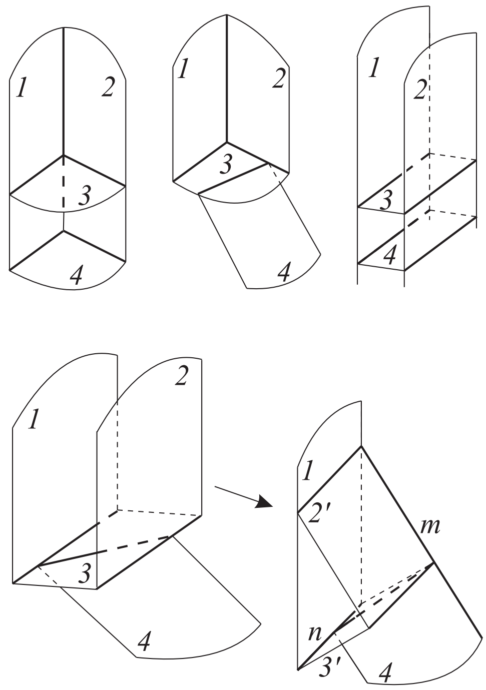

Figure 1.10: 4. case

   $3' \perp 4$. Then plane 3' is orthogonal to both plane 2' and plane 4 ⇒ plane 3' is orthogonal to their intersection line m. Let n be the intersection line of plane 1 and plane 3'. Then $n \perp m$ and $1 \circ 2 \circ 3 \circ 4 = 1 \circ 2' \circ 3' \circ 4$, but $2' \perp 3' \Rightarrow 1 \circ 2' \circ 3' \circ 4 = (1 \circ 3') \circ (2' \circ 4)$. Then this isometry is the composition of two rotation around orthogonal axis, therefore this is a screw displacement. ■